Mathematics Applied to Finance: Regularities in the VIX and the Distribution of Option's Payoff

A thesis submitted in partial fulfilment
of the requirements for the
Degree of Doctor of Philosophy
in Computational and Applied Mathematics

by Ulises Cárcamo Cárcamo

University of Canterbury

2004

ACKNOWLEDGEMENTS

This project was funded by Universidad EAFIT and by COLFUTURO.

It could not have been accomplished without the help of several people and institutions and I would like to thank them all.

I would like to thank my supervisors, Dr. Graeme Wake, Dr. Edwin Maberly and Dr. Don McNickle, for their support, advice and direction.
In particular to Professor Wake for his consistent encouragement and support regarding not only academic matters but also personal and familial ones.

To Professor Marberly for his suggestions, comments and guidance related to finance. Something was found during this research because he suggested where to look for it.

To Dr. McNickle for his support regarding resources and for his comments and suggestions.

I would like to thank my fourth supervisor Dr. Ian Coope. From the very beginning of this project he started the supervision of my writing skills and he always helped me in his fields of expertise.

I am also indebted with Dr. Marco Reale, always willing to give me a hand as advisor or with resources.

I am especially thankful for the support provided by the Department of Mathematics and Statistics, specially during this last year. Finishing this research is my way to say **Thank You!**.

I thank Mrs. Elizabeth Wake for her consistent help, during my stay in New Zealand, in personal and familial matters as well as Paul and Leigh Howells for help me and my family.

I also thank my family for their encouragement during the whole project.

Special thanks to the economist from the Universidad de Medellín, Pedro Nel Sierra Guerra. who helped me to discover my abilities for mathematics and logic, when I attended his course of geometry in 1974.

Pedro Nel: It was that geometry course what brought about my interest in logical and exact sciences.

ABSTRACT

This thesis presents three applications of Mathematics to Finance, from the empirical to the analytic level.

The first part shows how the CBOE's market volatility index (VIX) has seasonal movements, using several statistical and econometrical tools. These tools complement each other. The application is shown in a way that illustrates how dangerous it is to apply only ordinary least squares methods to look for seasonality.

The second part shows interesting patterns emerging from empirical distributions of S&P100 index (OEX) returns over some horizons conditioned to the VIX level.

The third part shows the main features of the distributions of option's payoff obtained using mainly analytical tools. However, a computer is needed to get a general picture of the distribution useful for speculators and traders in isolation.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

This thesis consists of three different projects. The first two are related to the Chicago Board Options Exchange (CBOE) Market Volatility Index (VIX) and the third one to option's payoff.

The first part is related to the seasonality in the VIX, the second part to the relationship VIX-OEX and the third part related to the distribution of the options payoff.

The methodologies vary from empirical to analytical, and several mathematical tools for deterministic and probabilistic phenomena are used.
It is intended to be not only a report on findings but also a guide for research in similar fields. This includes further research regarding the topics here presented.

1.1 The VIX

The CBOE's VIX has been interpreted as a measure of fear, anxiety, complacency and hope about how the stock market will behave in the future, especially during the next month following a given value of the index (Whaley 2000).

The stock market is subject to fluctuations of many types and many seasonal movements have been reported during the past two decades.

Since the market participants react to those fluctuations and seasonal movements, one should expect some of those movements reflected in the VIX. On the other hand, it is possible that this index has its own seasonality, independent, in some way, of those regularities observed in the stock indexes.
It is important to know about those regularities in case they exist and if it is possible, find out how to use them in get some economical or a theoretical reward (at least some knowledge about the behaviour of the market participants).

1.2 Seasonality in Stock Markets. A historical Perspective

The CAPM

The study of seasonality in the capital markets appeared related to the Capital Asset Pricing Model (CAPM) by Sharpe and Lintner, Sharpe (1964), Lintner (1965), and later extended by Merton (1969). This model, as many others, was developed as a model of competitive equilibrium [1] . The simplest version states that in a world without taxes the expected value of the yield of a risky asset is given by:

$$E(r_s) = r_f + \beta[E(r_m) - r_f] \tag{1.1}$$

Here: r_s is the yield (or return) of the risky asset (shares, for instance), r_f is the yield (or return) of a riskless asset (such as government bonds), r_m is the yield (or return) of the average risky asset (or market index), E is the expected value operator, and β is the relative risk measure for the risky asset.

[1] A market is competitive if it consists of a very large number of buyers and sellers that trade independently and in such a way that no one can significantly influence prices.
Equilibrium is a term which describes a situation in which economic agents or aggregates have no incentive to change their economic behaviour. A market is in equilibrium when, in the aggregate, buyers and sellers are satisfied with the current combination of prices and quantities bought or sold.

By definition

$$\beta = \frac{Cov(r_s, r_m)}{Var(r_m)} \tag{1.2}$$

where $Cov(r_s, r_m)$ is the covariance of r_s and r_m and $Var(r_m)$ is the variance of r_m.

The Efficient Markets Hypothesis

Although simple and elegant, the model expressed by (1.1) has not been exempt of problems. Conventional tests on the CAPM are in fact joint tests of (1.1) and the proposition that security markets are informationally efficient.

This last proposition evolved into the Efficient Markets Hypothesis (EMH). The use of ex-post asset returns in testing a model of equilibrium in capital markets implies of necessity that the observed returns in fact represent a series of informational equilibria; and conversely any attempt to test the informational efficiency of the markets will require a benchmark, which is provided by the equilibrium model of asset returns.

The EMH requires that capital market should be characterized by the lack of any ex-post regularities. If any of these existed, a market participant could use the regularity to devise a trading strategy that would yield above-normal returns. This would imply informational inefficiency.

Observed Anomalies: Regularities

The first regularities to be noted and documented for modern capital markets appeared in the work of Officer (1975), regarding the Australian share market, and in the work of Rozeff and Kinney (1976), regarding the market of United States. Following these works appeared a vast series of reports about anomalies on the joint hypotheses of the CAPM and the EMH. For instance, French (1980) reported anomalous behaviour on share market returns around period of non-trading at weekends; Banz (1981) reported an anomaly in the performance of equity returns when classified by firm size and Reinganum (1981) presented evidence on an anomaly based on earning yields.

Paradoxically, those anomalies appeared as market regularities that are not explained by theory or institutional practice. The turn-of-the year effect, the weekend effect and the small firm effect and the turn-of-the-month effect are among the most known. Although some of them were new others had existed in the market folklore for many years.

In the first half of the 1980's there was so much evidence and theoretical work on these regularities that a symposium was held in Brussels during December of 1985 to discuss the advances on the subject. Since then, every year the specialized journals present one or more articles on this interesting aspect of finance.

Trying to Take Advantage of the Anomalies

It is important to note that one thing is to discover an anomaly using series of data from the market and other very different one is to take advantage of it in a profitable way. Some strategies based on these regularities work very well in theory, but when some aspects of the real world such as transaction costs are considered, they loss their attractive.
Even more worrying is the fact that some anomalies seem to be induced by the process of data-mining: As soon as the researcher publishes his/her finding they vanished over the out-of-sample period.

Dimson and Marsh (1999) show how from 1955 up to 1987 smaller companies had outperformed the All Share Index (U.K) (Small firm effect), but after that period a reversal in the effect has been observed. A similar process has been observed in the U.S.

The following declaration by Richard Roll (1994) expresses the hopelessness that any researcher interested in this field should experience:

> Over the past decade, I have attempted to exploit many of the seemingly promising *inefficiencies* by actually trading significant amounts of money ... Many of these effects are surprisingly strong in the reported empirical work, but I have never yet found one that worked in practice.

1.3 Seasonality in the VIX. Preliminary Work

One of the first approaches to seasonality in the VIX appeared in Fleming et a. (1995). They worked on the differences in the period from 1996 to 1992, without the crash days, , and found some evidence of intra-week seasonal effects when the index is calculated on the basis of calendar days.[2] They used a regression model with dummy variables, without a constant and Hansen's heteroscedasticity method of moments estimation. After adjusting the calculations on the basis of trading days only Mondays seemed to present significant differences with respect to the other days.
They used these results to justify the calculation of the VIX using trading days instead of calendar days.

1.4 The Data

To check for regularities requires a good amount of information for two main reasons. First, the obtained results need to be significant and, second the asymptotic behaviour of the statistics employed in the study of the data can be assumed soundly.

For the purpose of this work, daily closing values for the VIX, from 1986 to 2002, were downloaded from the Yahoo Finance and the CBOE websites.

[2]The days 10/19/87 to 10/30/87 and 10/13/89 to 10/16/89 are excluded from that sample

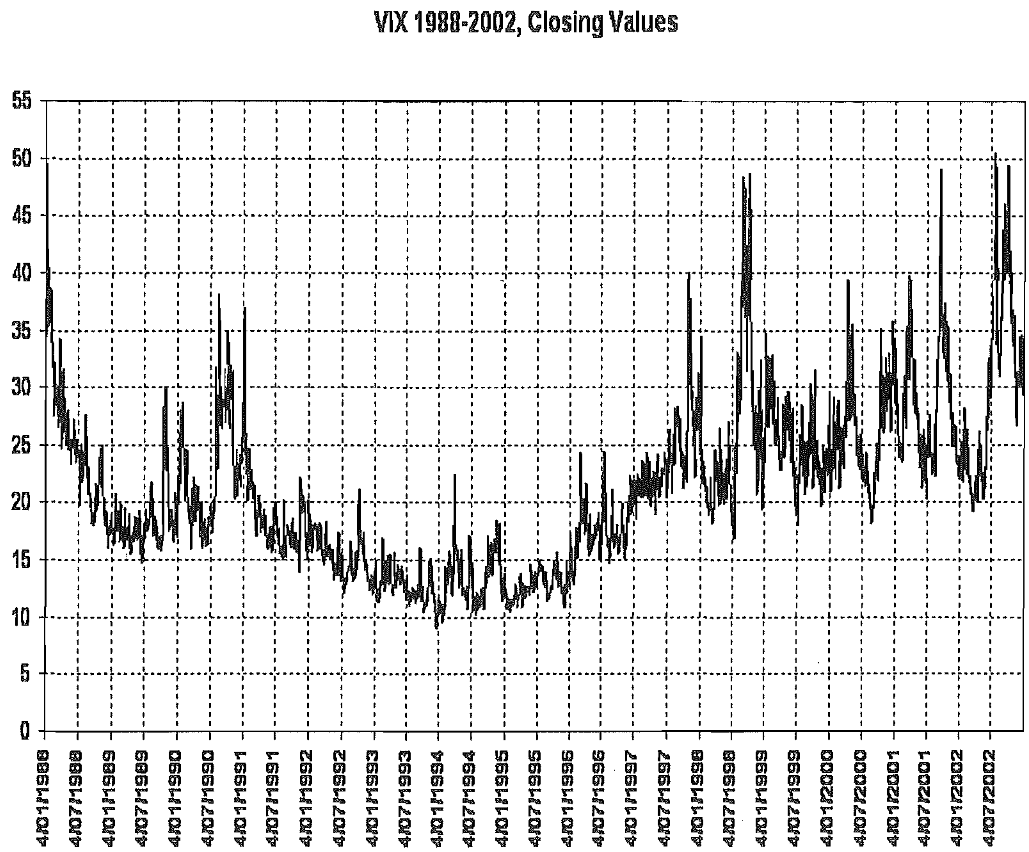


Figure 1.1: Restricted Set of Closing Values of the VIX

To avoid making adjustments to compensate for the effect of the 1987 market crash, the original data set was reduced by eliminating the first two years. The new series starts in January 1988 and has a total of 3,779 data points.

1.5 The Special Nature of the VIX

One of the most important characteristics of the VIX is the nature of its autocorrelations (see figures 1.2 and 1.3, and tables 1.1 and 1.2):

- Significant autocorrelations can be found at very high order lags.
- The first order partial autocorrelation is almost one.
- Most of the partial autocorrelations for lags from 2 to 12 are significant.
- At high lags significant partial autocorrelations are found more or less frequently.

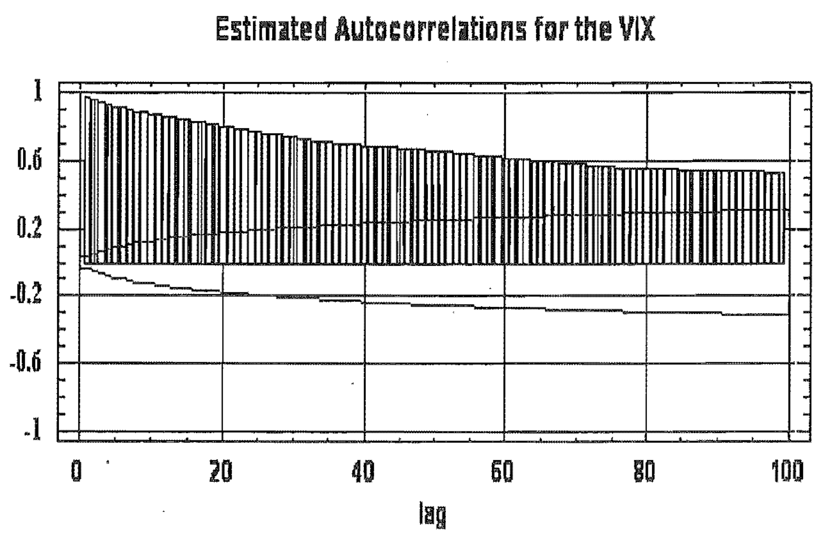


Figure 1.2: Autocorrelations and Partial Autocorrelations of the VIX

Preliminary tests gave negative results for unit roots. That is, the VIX is not a random walk although there is a very strong correlation between each

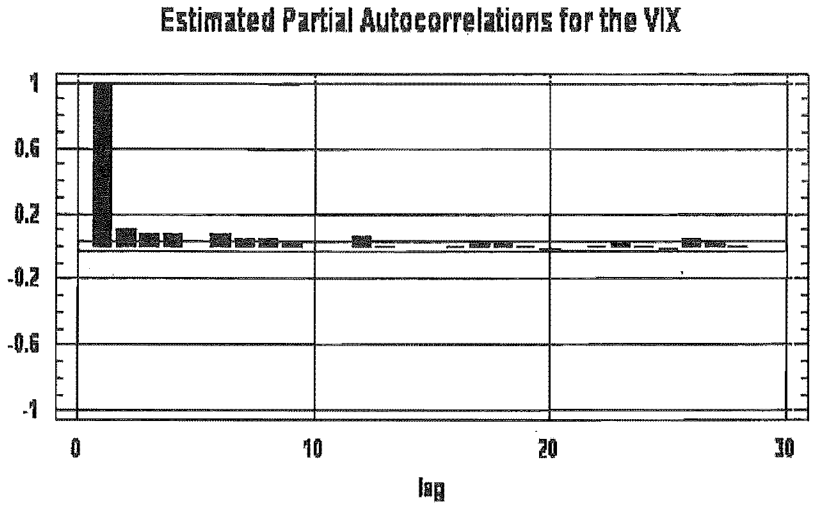


Figure 1.3: Autocorrelations and Partial Autocorrelations of the VIX

pair of consecutive values [3]

In addition, when higher order autocorrelations and partial autocorrelations are calculated and plotted, a complex pattern appears it is due possibly to the VIX's **mean reverting** nature (see figures 1.4 and 1.5).

The high autocorrelation of the VIX it is an obstacle to work directly with the index using elementary statistical and econometric tools. It is easier to work with the differences, $I_n - I_{n-1}$, quotients, $\frac{I_n}{I_{n-1}}$, or daily returns $ln(\frac{I_n}{I_{n-1}})$. By comparing tables 1.1 with table 1.3 and table 1. 5, and 1.2 with table 1.4 and table 1.6, it can be seen how the autocorrelation reduces dramatically in the differences and the returns. [4]

[3]Although the estimated autocorrelations of a random walk look similar to those of the VIX, the partial autocorrelations of such stochastic processes are negligible from the second lag on.

[4]The purpose of these tables and the autocorrelation and partial autocorrelation functions is only to show the reduction of the autocorrelation when working with the its differences or returns . There is no purpose of modelling these series as ARMA(p, q) series. Due to the

Table 1.1: Estimated Autocorrelations for the VIX, Selected lags

Lag	Autocorrelat	Stnd. Error	Low 95 %	Upp. 95%
1	0.977743	0.016267	-0.031883	0.031883
2	0.960341	0.027759	-0.054407	0.054407
3	0.945874	0.035478	-0.069535	0.069535
4	0.934206	0.041619	-0.081572	0.081572
5	0.921328	0.046841	-0.091806	0.091806
6	0.911751	0.051413	-0.100768	0.100768
7	0.903090	0.055527	-0.108831	0.108831
8	0.895908	0.059286	-0.116199	0.116199
9	0.889107	0.062767	-0.123021	0.123021
10	0.881113	0.066016	-0.129388	0.129388
50	0.658958	0.128342	-0.251546	0.251546
100	0.531584	0.159388	-0.312395	0.312395

This reduction allows the use of more simple techniques (some assuming statistical independence)in analyzing differences and returns than those needed to study the VIX series.

1.6 Possible Seasonalities

Because the data are daily close values, only regularities related to periods of more than one day can be studied: Weekly, monthly, semi-annually seasonalities and effects. For instance, monthly seasonality, *day-of-the- month effect*, *end-of-month effect* among others [5].

Initial tests gave no statistical evidence for the *month-of-the-year* in the differences nor in the returns.

Other tests gave negative results for a semi-annual type of seasonality regarding the periods May- October and November- April in both differences and returns. Statistics of the VIX level show, however, that there are more extreme values in the first period than in the second one.

The first part of the thesis pays attention to the *day-of-the-week-effect* in

persistent autocorrelation it will lead to high orders in p and q.

[5]With tic data it is possible to study intra-day seasonality

Table 1.2: Estimated Partial Autocorrelations for the VIX

Lag	Autocorrelat	Stnd. Error	Low 95 %	Upp. 95 %
1	0.977743	0.016267	-0.0318833	0.031883
2	0.099032	0.016267	-0.0318833	0.031883
3	0.070364	0.016267	-0.0318833	0.031883
4	0.073473	0.016267	- -0.0318833	0.031883
5	-0.01166	0.016267	-0.0318833	0.031883
6	0.076839	0.016267	-0.0318833	0.031883
7	0.037841	0.016267	-0.0318833	0.031883
8	0.048215	0.016267	-0.0318833	0.031883
9	0.033008	0.016267	-0.0318833	0.031883
10	-0.01554	0.016267	-0.0318833	0.031883
11	-0.01439	0.016267	-0.0318833	0.031883
12	0.054288	0.016267	-0.0318833	0.031883

Table 1.3: Estimated Autocorrelations for the VIX Differences

Lag	Autocorrelat	Stnd. Error	Low 95 %	Upp. 95 %
1	-0.11519	0.016269	-0.031887	0.031887
2	-0.07115	0.016484	-0.032308	0.032308
3	- 0.5654	0.016565	-0.032467	0.032467
4	0.001962	0.016616	-0.032567	0.032567
5	-0.060100	0.016616	-0.032567	0.032567
6	-0.025500	0.016673	-0.032679	0.032679
7	-0.039420	0.016684	-0.032700	0.032700
8	-0.06430	0.016708	-0.032748	0.032748
9	0.040906	0.016709	-0.032749	0.032749
10	0.023960	0.016736	-0.032801	0.032801

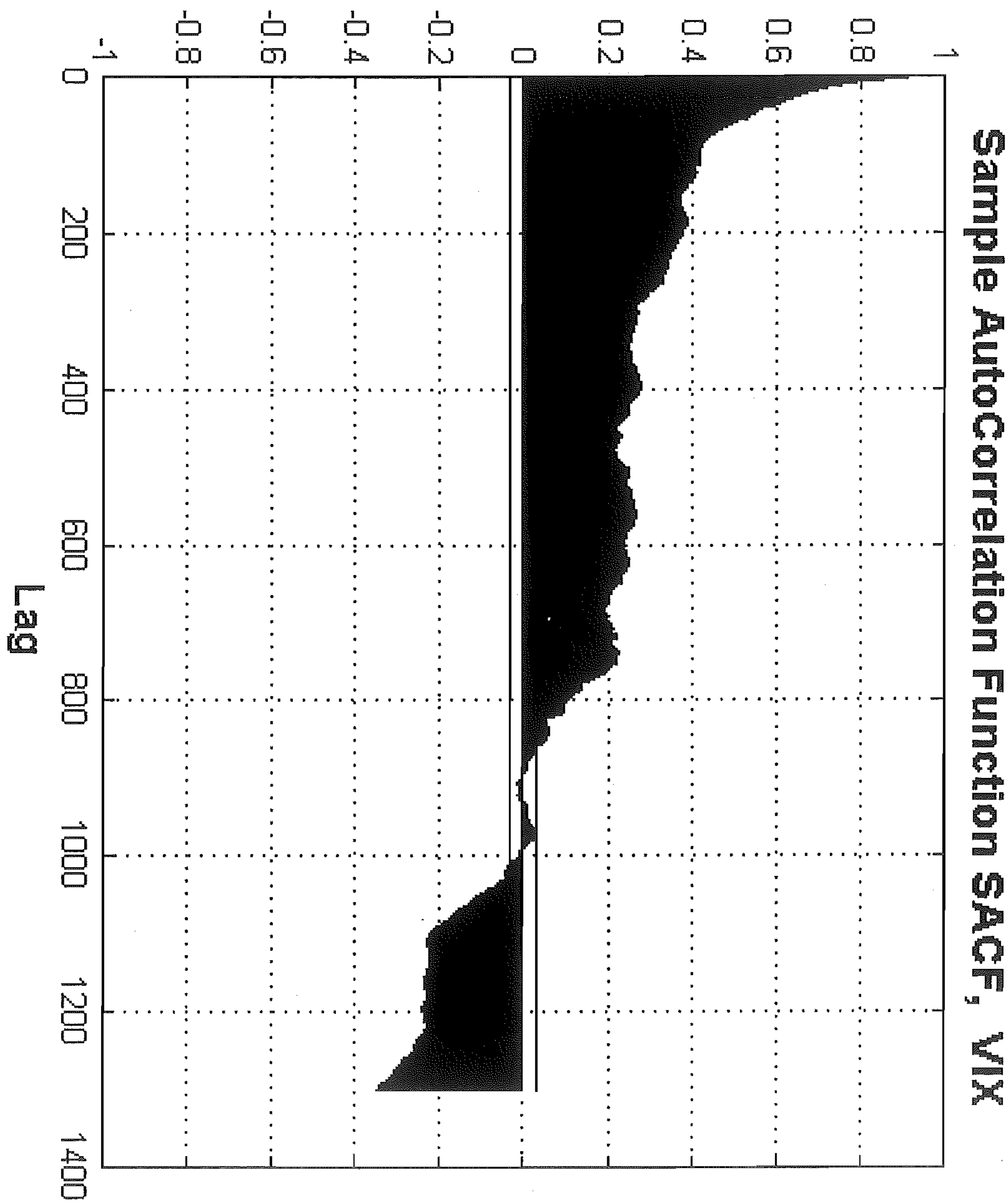


Figure 1.4: Autocorrelations for Very High Lags

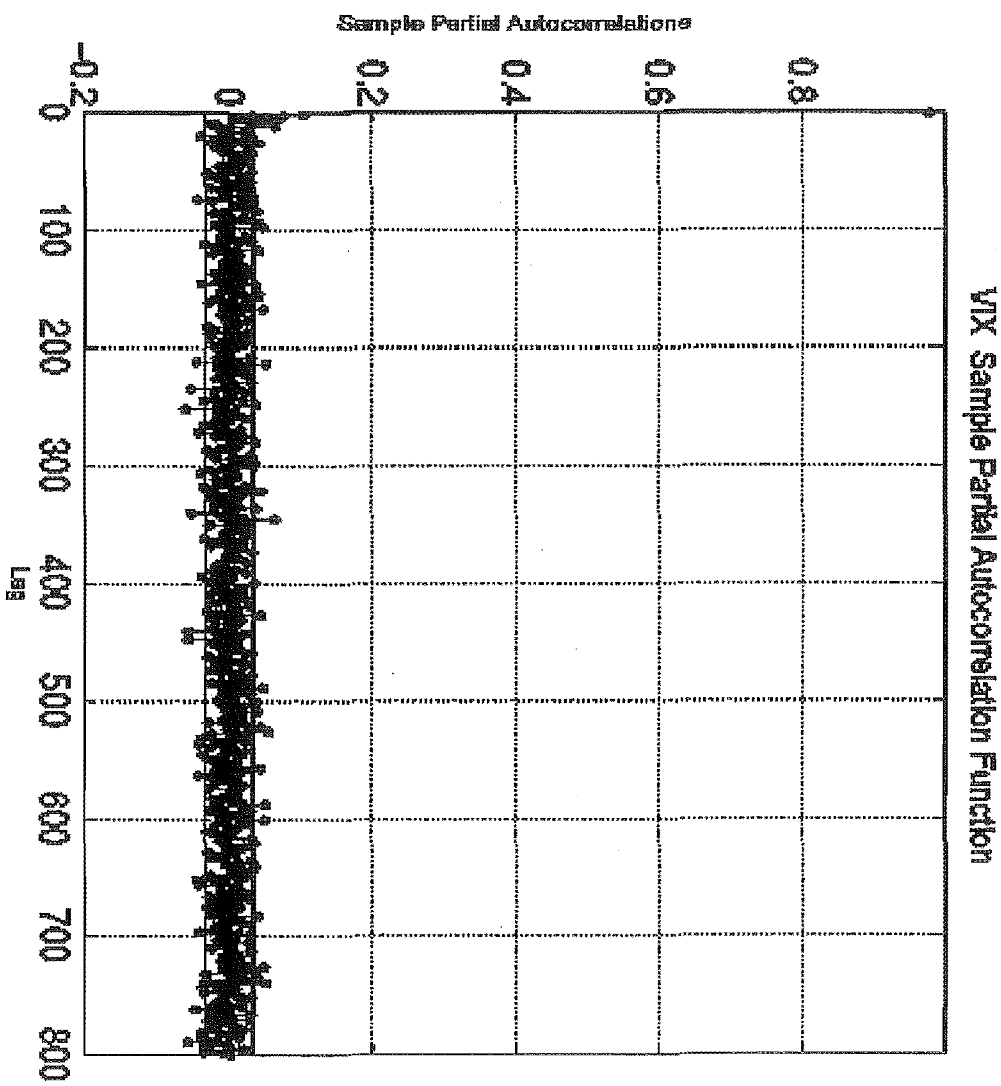


Figure 1.5: Partial Autocorrelations for Very High Lags

Table 1.4: Estimated Partial Autocorrelations for the VIX Differences

Lag	Autocorrelat	Stnd. Error	Low 95 %	Upp. 95 %
1	-0.11519	0.016262	-0.031887	0.031887
2	-0.08556	0.016262	-0.031887	0.031887
3	-0.07687	0.016262	-0.031887	0.031887
4	-0.02167	0.016262	-0.031887	0.031887
5	-0.07553	0.016262	-0.031887	0.031887
6	-0.05093	0.016262	-0.031887	0.031887
7	-0.06517	0.016262	-0.031887	0.031887
8	-0.03873	0.016262	-0.031887	0.031887
9	0.018168	0.016262	-0.031887	0.031887
10	0.015092	0.016262	-0.031887	0.031887

Table 1.5: Estimated Autocorrelations for the VIX Returns

Lag	Autocorrelat	Stnd. Error	Low 95 %	Upp. 95 %
1	-0.11783	0.016269	-0.031887	0.031887
2	-0.05454	0.016494	-0.032327	0.032327
3	- 0.07000	0.016541	-0.032420	0.032420
4	-0.01637	0.01662	-0.032574	0.032574
5	-0.04899	0.016624	-0.032582	0.032582
6	-0.01542	0.016662	-0.032657	0.032657
7	-0.02885	0.016684	-0.032700	0.032700
8	-0.01028	0.016679	-0.032690	0.032690
9	0.021169	0.016681	-0.032694	0.032694
10	0.015729	0.016688	-0.032707	0.032707

Table 1.6: Estimated Partial Autocorrelations for the VIX Returns

Lag	Autocorrelat	Stnd. Error	Low 95 %	Upp. 95 %
1	-0.11783	0.016269	-0.031887	0.031887
2	-0.06939	0.016269	-0.031887	0.031887
3	-0.08666	0.016269	-0.031887	0.031887
4	-0.04128	0.016269	-0.031887	0.031887
5	-0.06885	0.016269	-0.031887	0.031887
6	-0.04278	0.016269	-0.031887	0.031887
7	-0.05192	0.016269	-0.031887	0.031887
8	-0.03783	0.016269	-0.031887	0.031887
9	-0.00059	0.016269	-0.031887	0.031887
10	0.002704	0.016269	-0.031887	0.031887

both, differences and returns.

1.7 Tools for Studying Seasonality

The tools most widely used to study seasonality and anomalies can be classified into three groups: ANOVA-type methods, regression methods and time series models [6].

ANOVA-type Methods

In applying ANOVA-type methods, data belonging to each period (day, month, etc.) is grouped forming the so called *samples.* The main goal is to decide whether there are statistically significant differences among the group means or medians.

ANOVA assumes that the samples come from normal populations with equal variances and look for differences in the group means. Since financial data seldom behave as normal, this is not a good method to be used in testing seasonality in this case. In many cases, however, this method can shed some light on the seasonal structure and can be used as a first step, before using more

[6]Fourier analysis is also used with stationary time series

appropriate methods.

Non-parametric ANOVA-Type Methods

The nonparametric (or distribution-free) methods analogous to ANOVA do not assume any specific distribution from which the data come from. These methods compare medians instead of means. Beside this, variances are not relevant in this more general context so it is not necessary assume or check for equal variances. This set of characteristics makes these methods more suitable to deal with financial data.

The Kruskal-Wallis Test (Gultekin and Gultekin, 1983) and the Mood's Median Test are very useful to compare the samples when it is proper to assume independency of the samples. When this assumption is not admissible, the Friedmann's test should be used instead[7].

The Kruskal-Wallis, Mood and Friedman tests only show that, within certain confidence levels, there are differences among the medians, but they do not show which medians are statistically different. To find this it is necessary to apply the Multiple Comparisons Dunn's test that produces one statistic every possible pair of samples.

Regression Methods

Basic regression methods use seasonal dummy variables. For instance, if there are n seasons, in the Ordinary Least Squares (OLS) method (or , simply, Linear Regression Method), the regression equation

$$y_t = \alpha_0 + \alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_{n-1} D_{n-1} + \epsilon_t \tag{1.3}$$

[7] One of the uses of this test is documented in Wei (1996)

will be estimated. Here D_i is 1 in season i and 0 otherwise, for $i = 1, \cdots, n-1$, and the residuals ϵ_t are assumed independent and identically distributed with mean 0 and constant variance σ^2 (abridged as $\mathbf{iid}(\mathbf{0},\ \sigma^2)$).
The constant, α_0 is the mean value of the n-th sample . If μ_i is the mean of the i-th sample, then $\alpha_i = \mu_i - \alpha_0$.

In this model, the ϵ_t are assumed to be normally distributed and this assumption is used to find the distribution of the estimators of the α_i. The null hypothesis is that there are equal means ($\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$), that is, absence of seasonality.

When the variance cannot be assumed constant, a heteroscedastic-consistent estimator should be used. The White and the Davidson-Mackinnon estimators are of common use. If the independency assumption does not hold, new adjustments need to be done to the method. For instance, there is a modify regression method to apply when the residuals follow an ARMA model

$$\epsilon_t = \phi_0 + \sum_{k=1}^{p} \phi_k \epsilon_{t-k} + \eta_t + \sum_{k=1}^{q} \beta_k \eta_{t-k} \tag{1.4}$$

where η_t is white noise. There are also models for uncorrelated residuals that follow a GARCH model

$$\epsilon_t = \sigma_t \eta_t, \quad \sigma_t^2 = \alpha_0 + \sum_{k=1}^{p} \alpha_k \epsilon_{t-k} + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2, \tag{1.5}$$

also those models that include dummies in the conditional variance and some even more sophisticated.

The effect of a single period may be tested using the following regression scheme:

$$Y_t = \alpha_0 + \alpha_1 D + \epsilon_t, \tag{1.6}$$

where D is the dummy variable associated with the period. The null hypothesis to test is $\alpha_1 = 0$.

In any case it is important to check whether the assumptions of each model hold, at least approximately. Not taking this into account can lead to serious theoretical mistakes and erroneous conclusions.

Time Series Models

One of the simplest models for seasonal time series has the form

$$Y_t = \alpha_d Y_{t-d} + \epsilon_t, \quad t = 1, 2, \cdots, \tag{1.7}$$

where $Y_{-d+1}, Y_{-d+2}, \cdots, Y_0$ are initial conditions and the ϵ_t are iid(0, σ^2) random variables. In this model, monthly data are represented by d=12, quarterly data by d=4 and so on.

A more general model is the multiplicative one

$$(1 - \alpha_d B^d)(1 - \Theta_1 B - \cdots - \Theta_p B^p)Y_t = \epsilon_t \tag{1.8}$$

where B is the backshift operator defined by $B(Y_t) = Y_{t-1}, \epsilon_t$ is a sequance of iid(0, σ^2) random variables and $Y_0, Y_{-1}, \cdots, Y_{-p-d+1}$ are initial values. This model also assumes that all the roots of the polynomial

$$m^p - \Theta_1 m^{p-1} - \cdots - \Theta_p = 0 \tag{1.9}$$

are less than one in absolute value. This kind of models is used for data gathered on a seasonal basis, for instance quarterly, and assume that residual autocorrelation is insignificant. Since the VIX data correspond to daily values and the residuals are highly autocorrelated, these models were not used in this work.

Other Techniques and Models

There are other techniques used in modelling seasonality. Among them, structural models, Fourier analysis techniques as well as the so called PAR-PGARCH models (Frances and Paap 2000) that could deal with the regularities in the VIX closing values. The use of these tools in the study of the VIX seasonality is left for future research on robustness of the regularities found and the quest for new ones.

1.8 Other Relationships between the VIX and the OEX

Since the VIX is constructed from the value of options on the OEX, many aspects of the OEX behaviour are expected to influence the VIX. The opposite influence could be possible as well: Traders seeing the VIX as a market's consensus about how the market's behaviour will be in the immediate future, will modify their trading on blue-chip stocks and this will affect movements in the S&P100 index.

Looking for how to quantify this influence is an interesting research project that could shed some light on unknown links between the two indices. An even more interesting topic, from the point of view of practitioners, is the possibility of forecasting the OEX behaviour based on VIX. If this is not possible to study that possibility of forecast using the S&P100 index itself, then the possibility of forecast study using the OEX returns should be tried [8].

One initial purpose was to research about the possibility of forecasting the distribution of the OEX returns, over certain horizons, based on the VIX level. Two main results were obtained. First, in strict sense is not possible to speak about the distribution of the OEX returns over several horizons because of the high and persistent autocorrelation. Second, in spite of that, frequency distributions of the OEX returns, given the VIX level can be used to show certain regularities in the behaviour of the mean as well as the minimum return.

In particular, the behaviour of the minimum return given the VIX level is quite interesting for all horizons considered here. The methodology used in this

[8] Comparing the two indices directly could lead to a case of spurious correlation

part is basically empirical. The methods are explained and the results showed. So far any attempts that I have done to explain those results from an analytical point of view have been in vain. There are some mathematical tools related to the probability distribution of crossing barriers of a stochastic process but the theory has been well developed only for Gaussian (stationary) processes. Since the OEX is not Gaussian and presents autocorrelation, this theory is not appropriate to explain the phenomenon. In a further study of this phenomenon the latest results for non Gaussian processes will be studied and applied.

1.9 The Distribution of Option's Payoff

In the classical Black-Scholes (B-S) model, the price of the underlying stock is modelled as a Geometric Brownian motion, with two main parameters: drift and volatility. The payoff, at maturity, depends on those two factors, since the probability of the stock price to reach values greater than the strike value depend on them.
However options prices do not depend at all on the drift, the expected instantaneous (during a short interval of time, dt) rate of return per unit of time of the stock.

Since risk adverse people need to be offered a risk premium, an appropriate expected return, to take risk, the absence of the drift in the Black-Scholes formula implies that the option's price does not depend on any measure of risk aversion.

The B-S formula is said to be a risk neutral one, since risk neutral are people that do not need any premium to take risks and they make their decisions based on expected values.

The B-S model also assumes the existence of a risk free asset (a bond or a bank account) whose price evolves related to a deterministic interest rate r. The deduction of the formula needs the creation of a portfolio consisting of the stock and the risk free asset.

In the hypothetical situation in which an option is traded in isolation, no equivalent portfolio can be created.
On the other hand, it is the case a speculator who, based on his/her estimations about the volatility and drift of certain stock , suspects that, for a given horizon, the stock's drift will lead the option's price above the level given by the B-S model. He/she probably will want to make his/her own valuation of the option and for this purpose (that lead to some arbitrage) the B-S model is not useful. He/she will need the distribution of the payoff to calculate that valuation using whether the expected payoff or the probability that an option expires without value.
That distribution is also useful to estimate, for a given horizon, the implied drift that lead the option to expire out of the money.

The study of the most important characteristics of the distribution of the option's payoff is the subject of the third part of this thesis. The methodology used is mostly analytical. The assumptions of the Black-Scholes model are used to derive the main characteristics of the distribution. However, given the complexity of some of the expressions, computer programs are used in two main purposes. The first objective is to obtain some estimates of the drift and volatility, then to apply the results to real markets. The second objective is to construct some series of graphs that show a general picture of the behaviour of the probability of zero-payoff and the expected payoff.

Chapter 2

Day of the Week Effect

There is abundant evidence of the so called *weekend effect* and *day-of-the week effect* in the returns in several stock markets (see for instance, French 1980 and Lakonishok and Maberly 1990). Since market movements affect investor's sentiments, a natural question arises: Does the VIX have any of these effects?

2.1 Preliminary Evidence

Simple statistics show possible evidence of the *day-of-the-week-effect* in both differences and returns.

Table 2.1 shows that, on average, differences and returns are negative on Fridays and Mondays while they remain positive, on average, from Tuesday to Thursday.

Although Monday and Friday mean differences and returns are negative, they differ considerably: Friday's average difference is 12.11 times that of Monday's and Friday's average return is 24.34 times that of Monday's. Even more interesting is the fact that the absolute values of those means form an increasing, almost geometrically, sequence in both differences and returns (See figure 2.1). On the other hand, the standard deviations seem to be quite similar. At first

Table 2.1: **Means, Medians and Variances of Differences and Returns**

Means

	Mon	**Tue**	**Wed**	**Thu**	**Fri**
Diff	-0.018120	0.019974	0.063865	0.144487	-0.21950
Rets	-0.000430	0.002226	0.002596	0.005843	-0.01056

Medians

	Mon	**Tue**	**Wed**	**Thu**	**Fri**
Diff	-0.030000	0.040000	0.050000	0.050000	-0.260000
Rets	-0.001727	0.001679	0.002711	0.002901	-0.014481

Standard Deviations

	Mon	**Tue**	**Wed**	**Thu**	**Fri**
Diff	1.618795	1.37411	1.243045	1.473555	1.651981
Rets	0.062448	0.055639	0.050782	0.057989	0.068795
Observ.	718	773	771	760	756

glance those values show no significant differences, although Wednesdays seem to be the days of less variability. To really answer the question of the existence of these effects it is necessary to proceed with the statistical tests and the econometric models.

2.2 Day-of-the-Week Effect in the VIX Differences

2.2.1 Preliminary Statistical Analyses

Means

Since without a statistical analysis, data may be deceiving, some initial tests were run using the package Statgraphics. The first one was the Least Significant Differences (LSD) Test. The results showed that there are statistically significant differences between the following pairs of means: Monday- Thursday, Monday-Friday, Tuesday-Friday, Wednesday-Friday, and Thursday- Friday.

Variances

Three statistical tests to check whether the variances were equal were run: Cochran C-test, Bartlett's test and Hartley's test the results were:

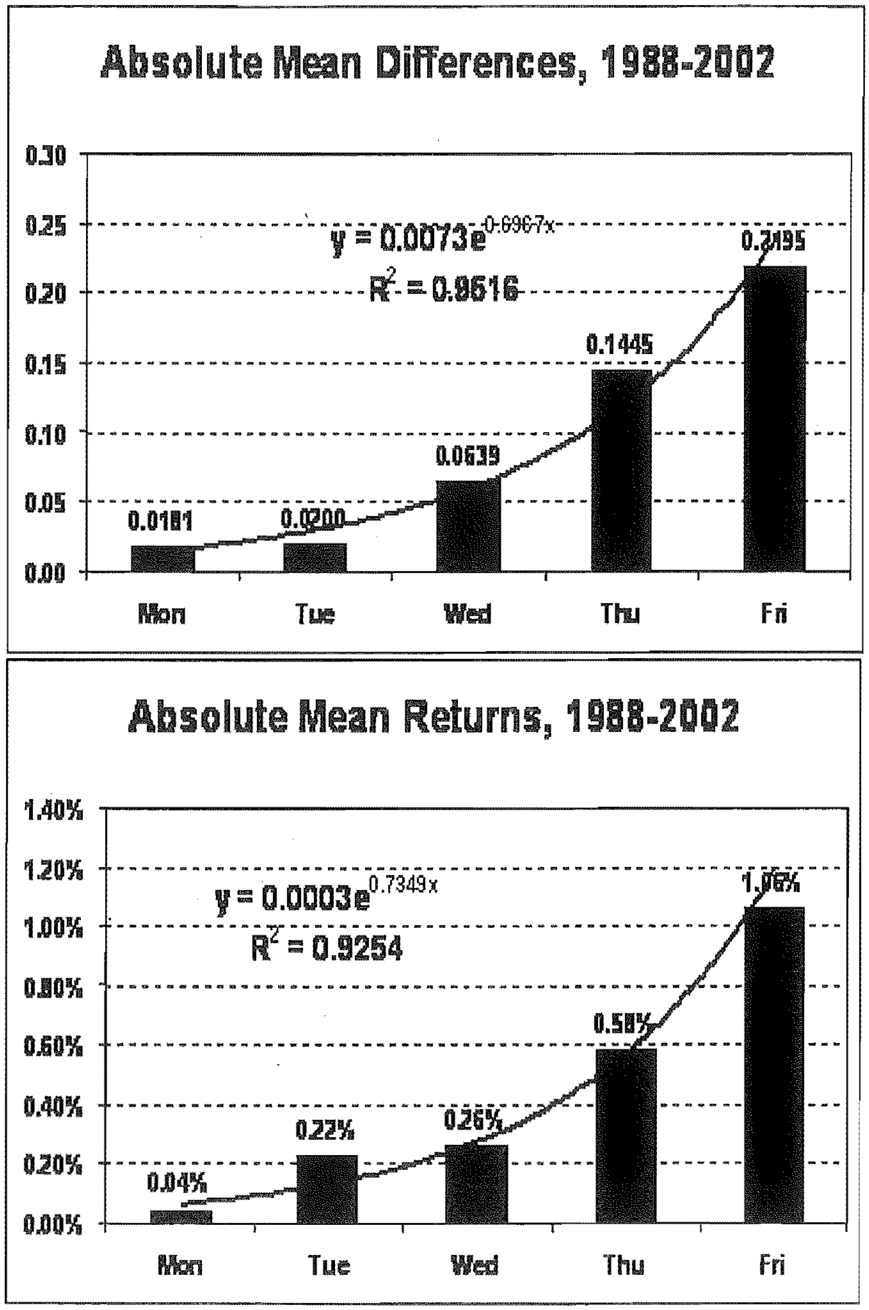


Figure 2.1: Daily Absolute Mean Differences and Returns

Cochran's C test: 0.249131, P-Value = 0.00000101841
Bartlett's test: 1.02194; P-Value = 0.0
Hartley's test: 1.76619

Under the assumption of normality these results mean that there are significant differences amongst the standard deviations at the 95.0% confidence level. This is a sign that even under such an assumption ANOVA should not be used.

Tests for Normality
The following tests for normality were run: Chi-Square goodness-of-fit, Shapiro-Wilks, Z score for skewness, Z score for kurtosis. Subsequently the respective P-values are calculated for each sample. With exception of the Z score for skewness for Wednesday, whose P-value reached 0.659801, all tests gave P-values less than 10^{-5}.

All this means that we can reject the hypothesis that every sample comes from a normal population at the 99% confidence. This reinforces the already reached conclusion that ANOVA should not be used in this analysis.

2.2.2 Result from the Kruskal-Wallis Test

The Kruskal-Wallis statistic was 54.3171 with a P-Value of 4.51653E-11. That means that, under the assumption that the correlation between samples is negligible, there are statistically significant differences amongst the medians at 95.0% confidence level..

2.2.3 Result from the Friedman's Test

Given the correlation already observed it is better to rely on the Friedman test than in the Kruskal-Wallis one. The only problem arises from the fact that Friedman's test requires samples with equal sizes. To solve this difficulty, random sub samples of size 700 were sampled from each group of data, having in mind to evaluate later, in some way, the effect of this procedure. On the other hand, since the very beginning of the project, it was planned to use at least two different procedures to detect the effect. In this way each one could reinforce the conclusions reached by using the other or contradict them forcing the revision of both methods.

The Friedman statistic was 33.8731. Since the critical values for a Chi-squared distribution with 5-1 = 4 degrees of freedom at respective levels of 95.0% and 99.0% are 12.592 and 16.812. It can be concluded at the two significance levels that there are significant differences among the medians.

2.2.4 Result from the Multiple Comparisons Dunn's Test

The procedure of the Dunn's test give the following table of T_{ij} statistics:

T_{ij} Dunn's Statistics. Day of the Week Effect in Differences

	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
Monday	0.0000	0.1352	0.5916	1.9439	3.6004
Tuesday	0.1352	0.0000	0.4564	1.8086	3.7356
Wednesday	0.5916	0.4564	0.0000	1.3522	4.1920
Thursdays	1.9439	1.8086	1.3522	0.0000	5.5442
Friday	3.6004	3.7356	4.1920	5.5442	0.0000

The following table shows the critical values for the joint significance levels and their respective individual levels. The null hypothesis of equal medians should be rejected, at such level, if the given T_{ij} statistic exceeds its critical value. α is the joint level, $\frac{\alpha}{20}$ is the individual level.

Critical Values for the T_{ij} at Five Different Levels

α	**0.05**	**0.10**	**0.15**	**0.20**	**0.25**
$\frac{\alpha}{20}$	0.00250	0.00500	0.00750	0.01000	0.01250
Z	2.80704	2.57583	2.43238	2.32635	2.24140

After comparing the last two tables, it is concluded that at the 99% confidence level the null hypothesis of equal medians is rejected for all those pairs containing Friday. This means 99% significant differences among those medians: A **Friday effect.**

2.2.5 Regression Methods

Regression with dummy variables is selected as a second methodology to check for significant differences between the means of Monday, Tuesdays, Wednesdays and Thursdays and that of Fridays. All regressions were run using the package EViews.

First Step: OLS to Check for the Friday Effect
As an initial step the linear regression model was run:

$$Dif_t = C + \alpha_1 Mon_t + \alpha_2 Tue_t + \alpha_3 Wed_t + \alpha_4 Thu_t + \epsilon_t \tag{2.1}$$

where Dif_t is the difference at time t and the residuals, ϵ_t, are independent and normally distributed.
The results are shown in the following table:

OLS Dummy Regression for the Differences

Variable	Coefficient	Std. error	Z-Statistic	Prob.
Mon	0.201378	0.076986	2.615780	0.0089
Tue	0.239471	0.075568	3.168958	0.0015
Wed	0.283362	0.075616	3.747371	0.0002
Thu	0.363984	0.075887	4.796411	0.0000
C	-0.2195	0.053731	-4.085130	0.0000

R-squared	0.006751	**Mean dependent var**	-0.00118
Adj. R-squared	0.005698	**S.D dependent var**	1.48158
S.E. of Regression	1.477353	**Akaike info criterion**	3.619703
Sum squared resid	8234.841	**Schwarz Criterion**	3.627957
Log likelihood	-6832.62	**F-statistic**	6.41137
Durbin-Watson stat	2.227727	**Prob (F-statistic)**	0.000039

This means that there are significant differences between the Friday's mean and each one of the other means at the 99% confidence level.

A residual analysis (not displayed here) shows significant autocorrelation and partial autocorrelation in the residuals and in the squared residuals. This is a signal that, in addition to the correlation, the series has ARCH effects. The linear regression model is not be trusted.

Another Model

Another regression model similar to that of equation 2.1 was run, but now ϵ_t was considered an ARMA process with ARCH effects in its own residuals. That is,

$$\epsilon_t = \phi_0 + \sum_{k=1}^{p} \phi_k \epsilon_{t-k} + \eta_t + \sum_{k=1}^{q} \theta_k \eta_{t-k} \tag{2.2}$$

with

$$\eta_t = \sigma_t h_t, \quad \sigma_t^2 = \alpha_0 + \sum_{k=1}^{p} \alpha_k \eta_{t-k} + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2, \tag{2.3}$$

where h_t is white noise. The coefficients were estimated using the Maximum Likelihood methodology with the Marquardt algorithm. The results are shown in the following table:

ML-ARCH Dummy Regression for the Differences				
Conditional Mean				
Variable	Coefficient	Std. error	Z-Statistic	Prob.
Mon	0.28052	0.068205	4.112892	0.0000
Tue	0.28052	0.081413	3.097009	0.0020
Wed	0.233606	0.07473	3.125993	0.0018
Thu	0.280808	0.074629	3.762748	0.0002
C	-0.21387	0.052144	-4.10151	0.0000

ML-ARCH Dummy Regr. Additional Information			
R-squared	0.050901	Mean dependent var	0.001495
Adj. R-squared	0.044703	S.D dependent var	1.440515
S.E. of Regression	1.407949	Akaike info criterion	3.203829
Sum squared resid	7285.027	Schwarz Criterion	3.24583
Log likelihood	-5902.08	F-statistic	8.212228
Durbin-Watson stat	1.978292	Prob (F-statistic)	0.000000

ARMA Coefficients of the Residuals				
Variable	Coefficient	Std. error	Z-Statistic	Prob.
ϕ_1	-0.14115	-0.141153	-7.30718	0.0000
ϕ_3	0.536394	0.057721	9.292891	0.0000
ϕ_{10}	0.035789	0.015926	2.247177	0.0246
ϕ_{25}	-0.03699	0.015137	-2.44391	0.0145
ϕ_{26}	-0.02790	0.013708	-2.03491	0.0419
ϕ_{32}	0.145312	0.032265	4.503700	0.0000
ϕ_{47}	0.041243	0.017244	2.391798	0.0168
ϕ_{50}	-0.04665	0.01749	-2.66713	0.0077
ϕ_{54}	-0.02025	0.011969	-1.69139	0.0908
ϕ_{63}	0.023710	0.012688	1.868679	0.0617
ϕ_{77}	-0.01813	0.010754	-1.68592	0.0918
ϕ_{78}	-0.01641	0.010762	-1.52514	0.1272

ARMA Coefficients of the Residuals (cont.)

Variable	Coefficient	Std. error	Z-Statistic	Prob.
θ_2	-0.11343	0.018968	-5.98010	0.0000
θ_3	-0.63199	0.053309	-11.8552	0.0000
θ_9	0.044302	0.017552	2.524113	0.0116
θ_{15}	-0.03885	0.013631	-2.84982	0.0044
θ_{32}	-0.17091	0.033370	-5.12181	0.0000
		Variance		
α_0	0.126384	0.054433	2.321813	0.0202
α_1	0.174978	0.044496	3.932394	0.0001
β_1	0.767647	0.044349	17.30913	0.0000

Here the ϕ terms are the Auto-Regressive ones, the θ terms are the Moving-Average ones, α_0, α_1 and β_1 are the coefficients of the GARCH(1, 1) model.

The high order Auto-Regressive terms that are significant show an effect of the characteristic structure of the VIX: Significant autocorrelation at very high lags.

Because of the values of the Z-statistics and their corresponding P-Values (in the "Prob" column), this more complicated model shows hat there are significant differences, at the 99% level, between the Friday mean and the means of the other week days. It is a second proof of a **Friday Effect** in the differences.

A Third Approach

A third approach to prove a Friday effect was made by using the model

$$Dif_t = \alpha_0 + \alpha_1 Fri_t + \epsilon_t, \tag{2.4}$$

where Fri is a dummy variable corresponding to Fridays. The restriction on the residuals to have constant variance is dropped. If there is no Friday effect $\alpha_1 = 0$, and this is the null hypothesis.

The following table shows the results from linear regression with heteroscedastic-consistent covariance matrix:

Linear Regression Results for Model 2.4

Variable	Coefficient	Std. Error	t-Statistic	Probability
Fri	-0.27293	0.065457	-4.16964	0.0000
α_0	0.053435	0.026032	2.052683	0.0402

R-squared	0.005433	**Mean dependent var**	-0.001181
Adj. R-squared	0.00517	**S.D dependent var**	1.48158
S.E. of Regression	1.477745	**Akaike info criterion**	3.619441
Sum squared resid	8245.767	**Schwarz Criterion**	3.622743
Log likelihood	-6835.124	**F-statistic**	20.62836
Durbin-Watson stat	2.225997	**Prob (F-statistic)**	0.000006

The results show evidence for the Friday effect even using the heteroscedasticity-consistent White covariance matrix: the very low P-value corresponding to the "Fri" variable is a signal that the Friday's mean is different from those of the other days.
In spite of this result, a more complete model was run. This one also takes in account ARMA and ARCH effects in the residuals (equations 2.2 and 2.3).

ML-ARCH Dummy Regression for the Friday Effect

Conditional Mean

Variable	Coefficient	Std. error	Z-Statistic	Prob.
Friday	-0.269998	0.060715	-4.447000	0.0000
α_0	0.0489100	0.013742	3.559292	0.0004

ARMA Coefficients of the Residuals

ϕ_1	0.732914	0.036663	19.99032	0.0000
ϕ_{19}	0.034032	0.019124	1.779567	0.0751
ϕ_{20}	-0.044476	0.020290	-2.19198	0.0284
ϕ_{25}	-0.043407	0.015463	-2.807210	0.0050
ϕ_{27}	0.023931	0.012220	1.958361	0.0502

ARMA Coefficients of the Residuals. (cont.)

ϕ_{34}	-0.041839	0.010666	-3.92273	0.0001
ϕ_{43}	0.016515	0.013730	1.202805	0.2291
ϕ_{50}	-0.029365	0.017153	-1.71196	0.0869
ϕ_{51}	0.061708	0.026316	2.344898	0.0190
ϕ_{52}	-0.041786	0.017295	-2.41609	0.0157
ϕ_{60}	0.015271	0.010938	-1.396223	0.1626
θ_2	-0.861269	0.024596	-35.01670	0.0000
Variance				
α_0	0.110507	0.055436	1.993434	0.0462
α_1	0.17538	0.044385	3.951363	0.0001
β_1	0.777311	0.041855	18.57167	0.0000

R-squared	0.050821	**Mean dependent var**	-2.42E-05
Adj. R-squared	0.046717	**S.D dependent var**	1.447847
S.E. of Regression	1.413622	**Akaike info criterion**	3.213583
Sum squared resid	7395.813	**Schwarz Criterion**	3.242027
Log likelihood	-5957.05	**F-statistic**	12.38492
Durbin-Watson stat	1.994566	**Prob (F-statistic)**	0.000000

This is another proof of the Friday effect in the differences.

2.3 Day-of-the-Week Effect in VIX Returns

2.3.1 Preliminary Statistical Analyses

Means

The same test run for the differences are run for the returns. The Least Significant Differences (LSD) test shows that there are statistically significant differences between these pair of means: Monday- Thursday; Monday-Friday; Tuesday-Friday; Wednesday-Friday: and Thursday- Friday (these coincide with those of the differences).

Variances
Cochran's, Bartlett's and Hartley's test are run to check the variances, their results are:

Cochran's C test: 0.267841, P-Value = 1.59373E-11
Bartlett's test: 1.02172; P-Value = 0.0
Hartley's test: 1.83519

Under the assumptions of normality these results mean that there are significant differences amongst the standard deviations at 95.0% confidence level. This is also a sign that even under the normality assumption ANOVA should not be used.

Tests for Normality
The following tests for normality are run: Chi-Square goodness-of-fit, Shapiro-Wilks, Z score for skewness, and Z score for kurtosis. Subsequently, the respective P-values are calculated for each sample. With the following exceptions all test give P-values less than 10^{-2}: Chi Square on Wed: $3.08E-01$,Shapiro-Wilks On Wed:0.308036, Shapiro-Wilks on Thu: 0.0875819, Z score for skewness on Thu: 0.55867 and Z score for skewness on Wed:0.233755.

Since for each sample the minimum P-Value is less than 0.01, we can reject the hypothesis that every sample comes from a normal population with 99% confidence. This also means that ANOVA should not be used in this analysis either.

2.3.2 Result from the Kruskal-Wallis Test

The Kruskal-Wallis statistic was 58.0468 with a P-Value of 7.46037E-12. These results mean that, under the assumption that the correlation between samples is negligible; there are statistically significant differences amongst the medians at 95.0% confidence level.

2.3.3 Result from the Friedman's Test

Random sub-samples of size 700 were generate out of each group of data.

The Friedman statistic is 57.1989. Since the critical values for a Chi-squared distribution with 5-1 = 4 degrees of freedom at respective levels of 95.0% and 99.0% are 12.592 and 16.812, it can be concluded at the two significance levels that there are significant differences among the returns medians.

2.3.4 Result from the Multiple Comparisons Dunn's Test

The procedure of the Dunn's test give the following table of T_{ij} statistics:

T_{ij} Dunn's Statistics. Day of the Week Effect in Returns

	Monday	Tuesday	Wednesday	Thursday	Friday
Monday	0.0000	2.9242	2.0284	2.6200	3.5158
Tuesday	2.9242	0.0000	0.8959	0.3043	6.4401
Wednesday	2.0284	0.8959	0.0000	0.5916	5.5442
Thursdays	2.6200	0.3043	0.5916	0.0000	6.1358
Friday	3.5158	6.4401	5.5442	6.1358	0.0000

Comparing this statistics with the table of critical values it can be concluded that the hypothesis of equal medians should be rejected at 99% level for all pairs containing Friday and for Monday - Tuesday and Monday- Thursday.

It is interesting to notice that although the results of the LSD test show a significant difference between the means of Monday and Thursday in both, dif-

ferences and returns, the non-parametric methods show the difference in the respective medians only in the returns.
This may be due to the different methodologies used, the objects being tested, the assumptions as well as the power to identify statistically significant differences of each test.

2.3.5 Regression Methods

Regression with dummy variables is also selected as a second methodology to check for a Friday Effect in the returns.

First Step: Linear Regression to Check for the Friday Effect in the Returns
The Ordinary Least Squares (OLS) regression model is run:

$$Ret_t = C + \alpha_1 Mon_t + \alpha_2 Tue_t + \alpha_3 Wed_t + \alpha_4 Thu_t + \epsilon_t \tag{2.5}$$

The results are shown in the following table:

OLS Dummy Regression for the Returns

Variable	Coefficient	Std. error	Z-Statistic	Prob.
Mon	0.010124	0.003094	3.272637	0.0011
Tue	0.012784	0.003037	4.209932	0.0000
Wed	0.013154	0.003039	4.328883	0.0000
Thu	0.016401	0.003049	5.378441	0.0000
C	-0.01056	0.002159	-4.88993	0.0000

R-squared	0.008898	Mean dependent var	-3.45E-05
Adj. R-squared	0.007847	S.D dependent var	0.0596
S.E. of Regression	0.059366	Akaike info criterion	-2.80887
Sum squared resid	13.29724	Schwarz Criterion	-2.80062
Log likelihood	5310.955	F-statistic	8.468037
Durbin-Watson stat	2.231618	Prob (F-statistic)	0.000001

This means that, under the assumptions of the linear regression (OLS) model, there are significant differences between the Friday's mean and the each one of the other means at the 99% level.

Another Model

Since here the residuals showed correlation and ARCH effects, another regression model similar to that of the differences is run.

ML-ARCH Dummy Regression for the Returns				
Conditional Mean				
Variable	**Coefficient**	**Std. error**	**Z-Statistic**	**Prob.**
Mon	0.012885	0.003443	3.742974	0.0002
Tue	0.015726	0.003318	4.73981	0.0000
Wed	0.014939	0.003201	4.667122	0.0000
Thu	0.016422	0.003224	5.093333	0.0000
C	-0.01234	0.002296	-5.37554	0.0000
ARMA Coefficients of the Residuals				
ϕ_1	-0.20966	0.046809	-4.479	0.0000
ϕ_2	0.722577	0.038375	18.8292	0.0000
ϕ_{21}	-0.01918	0.012958	-1.48018	0.1388
ϕ_{22}	-0.05092	0.01690	-3.0133	0.0026
ϕ_{24}	0.043493	0.019037	2.28462	0.0223
ϕ_{25}	-0.03651	0.01862	-1.96071	0.0499
ϕ_{26}	-0.06417	0.019352	-3.31601	0.0009
ϕ_{27}	0.032847	0.017601	1.866221	0.0620
ϕ_{28}	0.03262	0.015784	2.066624	0.0388
ϕ_{31}	-0.02121	0.014094	-1.50486	0.1324
ϕ_{34}	-0.05188	0.012506	-4.14853	0.0000
ϕ_{35}	-0.02465	0.013234	-1.86255	0.0625
θ_1	0.081437	0.041109	1.980993	0.0476
θ_2	-0.83592	0.036398	-22.9661	0.0000

Variance				
α_0	0.000559	0.00021	2.66713	0.0077
α_1	0.106866	0.026821	3.984387	0.0001
β_1	0.727661	0.064416	11.29636	0.0000

ML-ARCH Dummy Regr. Aditional Information			
R-squared	0.052501	**Mean dependent var**	1.96E-05
Adj. R-squared	0.047153	**S.D dependent var**	0.05926
S.E. of Regression	0.057846	**Akaike info criterion**	-2.912003
Sum squared resid	12.45118	**Schwarz Criterion**	-2.875399
Log likelihood	5471.813	**F-statistic**	9.818081
Durbin-Watson stat	2.023723	**Prob (F-statistic)**	0.000000

Again, this shows a Friday Effect in the returns.

A Third Approach

With the returns, a third approach to prove a Friday effect is also used. The model :

$$Ret_t = \alpha_0 + \alpha_1 Fri_t + \epsilon_t, \tag{2.6}$$

where Fri is a dummy variable corresponding to Fridays. The following table shows the results from OLS with heteroscedastic-consistent covariance matrix:

OLS Results for Model 2.4

Variable	**Coefficient**	**Std. Error**	**t-Statistic**	**Probability**
Fri	-0.013156	0.002706	-4.861892	0.0000
α_0	0.002598	0.001033	2.515255	0.0119

R-squared	0.007801	**Mean dependent var**	-3.45E-05
Adj. R-squared	0.007538	**S.D dependent var**	0.059600
S.E. of Regression	0.059375	**Akaike info criterion**	-2.809352
Sum squared resid	13.31195	**Schwarz Criterion**	-2.80605
Log likelihood	5308.866	**F-statistic**	29.68891
Durbin-Watson stat	2.230228	**Prob (F-statistic)**	0.000000

A more sophisticated model, which takes in account ARMA and ARCH effects in the residuals (Model 2.2, 2.3), is also run:

ML-ARCH Dummy Regression for the Friday Effect in the Returns				
Conditional Mean				
Variable	**Coefficient**	**Std. error**	**Z-Statistic**	**Prob.**
Frid	-0.014230	0.002803	-5.076123	0.0000
α_0	0.002713	0.000558	4.858821	0.0000
ARMA Coefficients of the Residuals				
ϕ_1	0.889609	0.017606	50.530000	0.0000
ϕ_4	0.021872	0.016066	1.3614227	0.1734
ϕ_9	0.032068	0.012171	2.634756	0.0084
ϕ_{12}	0.019399	0.011339	1.710745	0.0871
ϕ_{21}	-0.032990	0.012718	-2.593901	0.0095
ϕ_{23}	0.037770	0.014402	2.622481	0.0087
ϕ_{25}	-0.050475	0.015958	-3.162956	0.0016
ϕ_{27}	-0.051057	0.013565	3.763765	0.0002
ϕ_{30}	-0.017613	0.010822	-1.627443	0.1036
ϕ_{34}	-0.034339	0.016347	-2.100611	0.0357
ϕ_{35}	0.032273	0.024115	2.073262	0.0381
θ_2	-1.017136	0.024596	-42.179010	0.0000
θ_2	0.026915	0.023915	-1.125432	0.2604
Variance				
α_0	0.000604	0.000215	2.811105	0.0049
α_1	0.103882	0.027278	3.808266	0.0001
β_1	0.716070	0.067774	10.56554	0.0000

R-squared	0.054187	**Mean dependent var**	1.96E-05
Adj. R-squared	0.04987	**S.D dependent var**	0.05926
S.E. of Regression	0.057764	**Akaike info criterion**	-2.915198
Sum squared resid	12.42903	**Schwarz Criterion**	-2.885249
Log likelihood	5473.793	**F-statistic**	12.55346
Durbin-Watson stat	2.031037	**Prob (F-statistic)**	0.000000

This is more evidence of the Friday effect in the returns.

2.4 The Monday Effect

So far some evidence about a possible **Monday effect** has been found. The findings of Fleming et. Al (1995) , regarding the VIX differences during the period 1986-1992, and, the results for the returns from the Dunn's test.

However, when regression models that take into account ARMA and GARCH effects are run, none of them give positive results for such effect. A possible explanation for this is that, given a real difference between the means of Fridays and those of the other days, the first order autocorrelation and the conditional heteroscedasticity, can induce that effect on Mondays.

2.5 The Period 1993-2002

As a way of testing the robustness of this Friday effect, focusing on a period not previously considered, is to check for this effect during the periods of bull and bear markets in the last decade and the beginning of this one.

This is the period from 04/01/1993 to 31/12/2002 and contains 2, 514 data points.

The plan is to examine this period as a whole and then divide it into two sub periods: 1993-1999, **Bull Market Period**, and 2000-2002, **Bear Market Period**.

Descriptive Statistics for the Whole Sub-period

Descriptive Statistics for the Differences 1993-2002					
	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	0.0644	0.0240	0.0484	0.1609	-0.2598
Median	0.0600	0.0400	0.0400	0.0500	-0.2700
Standard Deviation	1.6909	1.5414	1.3339	1.5525	1.5668
Kurtosis	14.9257	7.8161	3.5032	4.7346	3.4046
Skewness	1.3847	-0.5623	-0.0061	1.0519	0.5198
Range	20.9100	17.5600	11.7800	13.8600	13.2900
Minimum	-7.1400	-9.5000	-5.9400	-5.3600	-5.0400
Maximum	13.7700	8.0600	5.8400	8.5000	8.2500
Observations	477	516	513	505	503

The table shows how the Friday's mean and median remain negative but increases in absolute value. On the other hand, unlike those of the whole period 1988- 2002, the Monday's mean and median are positive. This is observed for both differences and returns.

Descriptive Statistics for the Returns 1993-2002					
	Monday	Tuesay	Wednesday	Thursday	Friday
Mean	0.0032	0.0023	0.0016	0.0064	-0.0115
Median	0.0031	0.0017	0.0026	0.0025	-0.0145
Standard Deviation	0.0636	0.0606	0.0527	0.0598	0.0649
Kurtosis	5.3904	3.3184	0.9870	0.6917	4.0645
Skewness	0.6706	-0.0545	-0.1787	0.4002	0.9096
Range	0.6563	0.5548	0.3562	0.3890	0.5609
Minimum	-0.2338	-0.3230	-0.1958	-0.1443	-0.1927
Maximum	0.4225	0.2319	0.1604	0.2447	0.3683
Observations	477	516	513	505	503

2.5.1 Results of the Kruskal-Wallis Test

The Kruskal-Wallis statistic for the differences is 33.7075 that correspond to a P-value of 8.56E-07. The Kruskal-Wallis statistic for the returns is 35.2363 that correspond to a P-value of 4.15E-07. This means that, under the hypothesis of independence of the samples, there are significant differences between the medians in both differences and returns at the 99% confidence level.

2.5.2 Friedman and Dunn Tests for the Differences

A random sample of size 450 is generated for each day, the Friedman statistic is 36.6969. This means that significant differences exist among the medians at levels of 95% and 99%.

The statistics table for the Dunn's test is shown below:

T_{ij} **Dunn's Statistics. Day of the Week Effect in Differences. 93-02**

	Monday	Tuesday	Wednesday	Thursday	Friday
Monday	0.0000	1.1173	0.1054	1.5390	5.2494
Tuesday	1.1173	0.0000	1.0119	0.4216	4.1320
Wednesday	0.1054	1.0119	0.0000	1.4336	5.1440
Thursdays	1.5390	0.4216	1.4336	0.0000	3.7104
Friday	5.2494	4.1320	5.1440	3.7104	0.0000

This means that Friday's median is different from those of the other days at a 99% level.

2.5.3 Friedman and Dunn Tests for the Returns

A random sample of size 450 is generated for each day, the Friedman statistic was 28.151. This means significant differences exist among the medians at levels of 95% and 99%.

The statistics table for the Dunn's test is shown below. The results of the table show that not only that Friday's returns median is different from those of the other days at a 99% level, but also that there is a difference between the medians of Mondays and Thursdays at 95% level.

T_{ij} Dunn's Statistics. Day of the Week Effect in Returns. 93-02

	Monday	Tuesday	Wednesday	Thursday	Friday
Monday	0.0000	0.6746	1.9606	2.2979	2.4033
Tuesday	0.6746	0.0000	1.2860	1.6233	3.0780
Wednesday	1.9606	1.2860	0.0000	0.3373	4.3639
Thursdays	2.2979	1.6233	0.3373	0.0000	4.7013
Friday	2.4033	3.0780	4.3639	4.7013	0.0000

2.5.4 Regression Models

Two basic linear regression models, of types 1.3 and 1.6, with White heteroscedasticity-consistent covariance matrix are run. They exhibit significant differences among Friday mean and the other day's means. Then other models containing ARMA and GARCH effects in the residuals are run. Both exhibit significant differences in the means of Fridays and those of the other days.

ML-ARCH Dummy Regression for the Differences 93-02

Conditional Mean

Variable	Coefficient	Std. error	Z-Statistic	Prob.
Mon	0.343265	0.077139	4.449981	0.0000
Tue	0.311812	0.074062	4.210154	0.0000
Wed	0.296141	0.070169	4.220401	0.0000
Thu	0.311421	0.065276	4.770822	0.0000
C	-0.257547	0.048683	-5.290268	0.0000

ARMA Coefficients of the Residuals				
ϕ_1	-0.136388	0.020177	-6.759504	0.0000
ϕ_2	0.647360	0.053852	12.021180	0.0000
ϕ_{16}	-0.040669	0.014613	-2.782972	0.0054
ϕ_{21}	-0.025674	0.013677	-1.877154	0.0605
ϕ_{24}	0.026844	0.017720	1.514878	0.1298
ϕ_{26}	-0.035679	0.018908	1.886954	0.0592
ϕ_{34}	-0.044770	0.013139	-3.407279	0.0007
ϕ_{50}	-0.055059	0.016011	-3.438954	0.0006
ϕ_{51}	0.037693	0.018719	2.013652	0.0440
ϕ_{53}	-0.038919	0.017870	-2.177846	0.0294
θ_1	-0.725136	0.049665	-14.600610	0.0000
Variance				
α_0	-0.017352	0.021387	-0.811297	0.4172
α_1	0.227157	0.043535	5.217841	0.0000
β_1	0.793299	0.029412	26.97239	0.0000
γ	0.260172	0.131723	1.975143	0.0483

ML-ARCH Dummy Regr. Diff. 93-02. Additional Information			
R-squared	0.051554	**Mean dependent var**	0.007087
Adj. R-squared	0.044172	**S.D dependent var**	1.555673
S.E. of Regression	1.520926	**Akaike info criterion**	3.243812
Sum squared resid	5646.562	**Schwarz Criterion**	3.291015
Log likelihood	-3971.511	**F-statistic**	6.983371
Durbin-Watson stat	1.977952	**Prob (F-statistic)**	0.000000

In this model it is necessary to introduce the dummy variable *Fri*, with coefficient γ in the variance equation to improve the approximation, reducing correlation and ARCH effect in the residuals.

The conditional variance takes the form:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \beta_1 \epsilon_{t-1}^2 + \gamma Fri \tag{2.7}$$

A similar model is run for the returns:

ML-ARCH Dummy Regression for the Returns 93-02

Conditional Mean

Variable	Coefficient	Std. error	Z-Statistic	Prob.
Mon	0.017263	0.004159	4.150752	0.0000
Tue	0.016057	0.003756	4.275366	0.0000
Wed	0.015338	0.003615	4.242992	0.0000
Thu	0.015915	0.003718	4.280225	0.0000
C	-0.013093	0.002641	-4.958283	0.0000
ARMA Coefficients of the Residuals				
ϕ_2	0.746021	0.047410	15.735680	0.0000
ϕ_5	-0.268791	0.056172	-4.785181	0.0000
ϕ_6	-0.035491	0.022249	-1.595183	0.1107
ϕ_{14}	-0.050744	0.013728	-3.696469	0.0002
θ_1	-0.124521	0.016574	-7.512939	0.1298
θ_2	-0.795200	0.047701	-16.670340	0.0000
θ_5	0.275259	0.042184	6.525163	0.0000
Variance				
α_0	0.000103	0.000106	0.967528	0.3333
α_1	0.119531	0.027470	4.351361	0.0000
β_1	0.798241	0.043004	18.562120	0.0000
γ	0.001012	0.000469	2.156585	0.0310

ML-ARCH Dummy Regr. Diff. 93-02. Aditional Information

R-squared	0.049812	Mean dependent var	0.0004
Adj. R-squared	0.044074	S.D dependent var	0.0607
S.E. of Regression	0.059341	Akaike info criterion	-2.8845
Sum squared resid	8.746963	Schwarz Criterion	-2.8472
Log likelihood	3621.59000	F-statistic	8.6813
Durbin-Watson stat	2.022697	Prob (F-statistic)	0.000000

These results show how the Friday effect is stronger in this sub-period than in the whole period considered before.

2.6 Differences and Returns during the Bull Market Period

2.6.1 Descriptive Statistics

The descriptive statistics of differences show significant changes with respect to those of the whole period. The mean and median for Monday is now positive, while those of Tuesday are now negative. On the other hand, the Wednesdays mean is negative while the median is positive. Thursdays mean and median remain positive and Friday mean and median remain not only negative but their values are close to those of Fridays in the sub period 93-02.

Descriptive Statistics for the Differences 1993-1999

	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	0.0797	-0.0118	-0.0111	0.2499	-0.2670
Median	0.0700	-0.0200	0.0200	0.0900	-0.1900
Standard Deviation	1.5319	1.3781	1.1571	1.4583	1.4365
Kurtosis	21.3216	14.0525	5.1051	7.1023	5.9841
Skewness	2.1194	-1.7318	-0.1094	1.3923	0.8055
Range	19.5000	16.3500	11.7800	13.8600	13.2400
Minimum	-5.7300	-9.5000	-5.9400	-5.3600	-4.9900
Maximum	13.7700	6.8500	5.8400	8.5000	8.2500
Observations	335	363	361	355	352

The descriptive statistics of the returns also show that means and medians are positive on Monday and Thursday, and negative on Friday. On Tuesday, the mean is positive and the median negative. On Wednesdays the signs swap.

Descriptive Statistics for the Returns 1993-1999

	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	0.0040	0.0012	-0.0007	0.0106	-0.0131
Median	0.0044	-0.0011	0.0012	0.0055	-0.0148
Standard Deviation	0.0644	0.0600	0.0522	0.0611	0.0666
Kurtosis	6.0999	4.6160	1.1928	0.8495	5.2928
Skewness	0.7753	-0.3057	-0.2918	0.3845	1.1731
Range	0.6563	0.5548	0.3562	0.3890	0.5439
Minimum	-0.2338	-0.3230	-0.1958	-0.1443	-0.1756
Maximum	0.4225	0.2319	0.1604	0.2447	0.3683
Observations	335	363	361	355	352

2.6.2 Results of the Kruskall-Wallis and Dunn's Tests

The Kruskall-Wallis statistic for the Differences is 34.2095 with a P-value of 6.74989E-7.For the returns the statistic is 36.7908 with a P-value Of 1.9891E-7. This means that, under independence assumptions, the medians are significantly different in both differences and returns at 99% level.

2.6.3 Results of the Friedman Test

The Friedman's test statistic for the differences was 17.8480. For the returns it is 46.5093.This means that the medians are significantly different in both differences and returns at 99% level.

T_{ij} Dunn's Statistics. Day of the Week Effect. Differences. 93-99

	Monday	Tuesday	Wednesday	Thursday	Friday
Monday	0.0000	1.2652	0.5422	0.2066	2.8402
Tuesday	1.2652	0.0000	1.8074	1.4717	4.1054
Wednesday	0.5422	1.8074	0.000	0.3357	2.298
Thursdays	0.2066	1.4717	0.3357	0.0000	2.6336
Friday	2.8402	4.1054	2.298	2.6336	0.0000

The Dunn's statistics table for the returns shows that, in addition to a Friday effect, there seems to be a Thursday effect. In fact there are also significant differences between the medians of Thursdays and those of Mondays, Tuesdays and Wednesdays.

T_{ij} Dunn's Statistics. Day of the Week Effect. Returns. 93-99

	Monday	Tuesday	Wednesday	Thursday	Friday
Monday	0.0000	0.3615	1.1103	2.6078	4.1054
Tuesday	0.3615	0.0000	0.7488	2.9693	3.7439
Wednesday	1.1103	0.7488	0.000	3.7181	2.9951
Thursdays	2.6078	2.9693	3.7181	0.0000	6.7132
Friday	4.1054	3.7439	2.9951	6.7132	0.0000

2.6.4 Regression Models

Two Initial linear regression models, with White heteroscedasticity-consistent covariance matrices, of respective types 1.3 and 1.6 are run. The results show that the Friday effect is strong in this set of data.

OLS Heterosc.Consist. Friday Effect. Differences. 93-99				
Variable	Coefficient	Std. Error	t-Statistic	Probability
Mon	0.346690	0.113428	3.056488	0.0023
Tue	0.255170	0.105329	2.422608	0.0155
Wed	0.255908	0.097833	2.615758	0.0090
Thu	0.516932	0.108870	4.748140	0.0000
C	-0.266989	0.076564	-3.487141	0.0005
R-squared	0.014194	Mean dependent var		0.007452
Adj. R-squared	0.011955	S.D dependent var		1.404124
S.E. of Regression	1.395706	Akaike info criterion		3.507504
Sum squared resid	3430.417	Schwarz Criterion		3.52301
Log likelihood	-3092.126	F-statistic		6.339095
Durbin-Watson stat	2.267982	Prob (F-statistic)		0.000046

OLS Heterosc.Consist. Friday Effect. Returns. 93-99				
Variable	Coefficient	Std. Error	t-Statistic	Probability
Mon	0.017170	0.004999	3.434762	0.0006
Tue	0.014361	0.004743	3.027955	0.0025
Wed	0.012406	0.004489	2.763799	0.0058
Thu	0.023763	0.004808	4.942011	0.0000
C	-0.013138	0.003548	-3.702419	0.0002
R-squared	0.016037	Mean dependent var		0.000384
Adj. R-squared	0.013802	S.D dependent var		0.061413
S.E. of Regression	0.060987	Akaike info criterion		-2.75347
Sum squared resid	6.549995	Schwarz Criterion		-2.73796
Log likelihood	2436.313	F-statistic		7.175553
Durbin-Watson stat	2.265171	Prob (F-statistic)		0.00001

Two regression models of types 1.3 and 1.6, respectively, but with White Heteroscedasticity- consistent, covariance matrix and ARMA and ARCH effects in the residuals are run (1.4 and 1.5). Their results confirm the Friday effect in this sub period of bull market in the differences as well as in the returns.

ML-ARCH Dummy Regression. Friday Effect. Differences 93-99				
Conditional Mean				
Variable	Coefficient	Std. error	Z-Statistic	Prob.
Frid	-0.336427	0.063500	-5.298057	0.0000
α_0	0.068701	0.012587	5.458208	0.0000
ARMA Coefficients of the Residuals				
ϕ_1	-0.149495	0.032418	-4.611468	0.0000
ϕ_3	-0.082436	0.021183	-3.891588	0.0001
ϕ_4	0.756909	0.038550	19.634720	0.0000
ϕ_5	0.122556	0.037327	3.283310	0.0010
ϕ_8	0.067817	0.026223	2.586134	0.0097
ϕ_{15}	0.058336	0.018086	3.225512	0.0013
ϕ_{22}	-0.059568	0.016170	-3.683766	0.0002
θ_4	-0.872409	0.028466	-30.647770	0.0000
θ_5	-0.063365	0.025893	-2.447201	0.0144
Variance				
α_0	-0.028698	0.020126	-1.425889	0.1539
α_1	0.237051	0.056983	4.160006	0.0000
β_1	0.784576	0.037523	20.909150	0.0000
γ	0.305421	0.136636	2.235296	0.0254

R-squared	0.054698	**Mean dependent var**	0.008641
Adj. R-squared	0.047044	**S.D dependent var**	1.411187
S.E. of Regression	1.377593	**Akaike info criterion**	2.978724
Sum squared resid	3281.23	**Schwarz Criterion**	3.025718
Log likelihood	-2582.447	**F-statistic**	7.14613
Durbin-Watson stat	2.026389	**Prob (F-statistic)**	0.000000

ML-ARCH Dummy Regression. Friday Effect. Returns 93-99				
Conditional Mean				
Variable	Coefficient	Std. error	Z-Statistic	Prob.
Frid	-0.017749	0.003856	-4.603356	0.0000
α_0	0.003748	0.000946	3.960456	0.0001
ARMA Coefficients of the Residuals				
ϕ_2	0.888131	0.050002	17.761800	0.0000
ϕ_4	-0.327679	0.061419	-5.335145	0.0000
ϕ_{11}	-0.037562	0.019948	-1.883036	0.0597
ϕ_{14}	-0.029039	0.017453	-1.663812	0.0962
θ_1	-0.126494	0.019309	-6.550994	0.0000
θ_2	-0.925584	0.042257	-21.903830	0.0000
θ_4	0.313029	0.052953	5.911436	0.0000
Variance				
α_0	0.000384	0.000130	2.954062	0.0031
α_1	0.123786	0.038208	3.239767	0.0012
β_1	0.774034	0.059885	12.925410	0.0000
R-squared	0.059883	Mean dependent var		0.000458
Adj. R-squared	0.053940	S.D dependent var		0.061478
S.E. of Regression	0.059797	Akaike info criterion		-2.866266
Sum squared resid	6.221587	Schwarz Criterion		-2.828811
Log likelihood	2522.849000	F-statistic		10.075850
Durbin-Watson stat	2.067239	Prob (F-statistic)		0.000000

2.6.5 Thursday Effect

After the Dunn's test results, two simple linear regression models with White heteroscedastic-consistent covariance matrices are run as a first step in a second way to prove the existence of the effect. The results show significant differences between the means of Thursday and those of Tuesdays and Wednesdays (in addition to that of Fridays).

OLS Heterosc.Consist. for Thursday Effect. Differences. 93-99

Variable	Coefficient	Std. Error	t-Statistic	Probability
Mon	-0.170242	0.113994	-1.493434	0.1355
Tue	-0.261762	0.105938	-2.470889	0.0136
Wed	-0.261024	0.098489	-2.650281	0.0081
Fri	-0.516932	0.108870	-4.748140	0.0000
C	0.249944	0.077400	3.229243	0.0013

R-squared	0.014194	Mean dependent var	0.007452
Adj. R-squared	0.011955	S.D dependent var	1.404124
S.E. of Regression	1.395706	Akaike info criterion	3.507504
Sum squared resid	3430.417	Schwarz Criterion	3.52301
Log likelihood	-3092.126	F-statistic	6.339095
Durbin-Watson stat	2.267982	Prob (F-statistic)	0.000046

OLS Heterosc.Consist. for Thursday Effect. Returns. 93-99

Variable	Coefficient	Std. Error	t-Statistic	Probability
Mon	-0.006593	0.004788	-1.376982	0.1687
Tue	-0.009402	0.004520	-2.079838	0.0377
Wed	-0.011357	0.004253	-2.670249	0.0076
Fri	-0.023763	0.004808	-4.942011	0.0000
C	0.010626	0.003245	3.274493	0.0011

R-squared	0.016037	Mean dependent var	0.000384
Adj. R-squared	0.013802	S.D dependent var	0.061413
S.E. of Regression	0.060987	Akaike info criterion	-2.75347
Sum squared resid	6.549995	Schwarz Criterion	-2.73796
Log likelihood	2436.313	F-statistic	7.175553
Durbin-Watson stat	2.265171	Prob (F-statistic)	0.00001

However, when regression models that take in account ARMA and ARCH effects in the residuals are run, the significance of the above mentioned differences disappear.

Again, it seems that this effect, as well as the observed effect on Mondays, is more related to the autocorrelation of the residuals and the squared residuals than to a real cause of the market. In fact this is an example of why one should be cautious with the results of linear regression or least squares with heteroscedastic-consistent covariance matrix alone.

2.7 Results During the Bear Market

2.7.1 Descriptive Statistics

Descriptive Statistics. Differences 2000-2002

	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	0.0282	0.1089	0.1896	-0.0497	-0.2431
Median	0.0350	0.1600	0.1900	-0.1000	-0.3600
Std. Deviation	2.0229	1.8753	1.6770	1.7427	1.8404
Kurtosis	7.9434	1.7896	1.4994	1.6722	0.5533
Skewness	0.5826	0.5184	-0.0584	0.6969	0.1696
Range	18.2100	12.5000	10.6300	10.6100	10.4200
Minimum	-7.1400	-4.4400	-5.5300	-4.3300	-5.0400
Maximum	11.0700	8.0600	5.1000	6.2800	5.3800
Observations	142	153	152	150	151

Descriptive Statistics. Returns 2000-2002

	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	0.001108	0.004728	0.007212	-0.003459	-0.007686
Median	0.001274	0.005986	0.006160	–0.004180	-0.014421
Std. Deviation	0.061711	0.062123	0.053507	0.055311	0.060761
Kurtosis	3.487620	0.591727	0.485226	0.037009	0.238672
Skewness	0.383752	0.479031	0.053649	00.361610	0.139966
Range	0.483130	0.336301	0.326286	0.307105	0.341806
Minimum	-0.200332	-0.133247	-0.176374	-0.142983	-0.192684
Maximum	0.282799	0.203055	0.149911	0.164122	0.149122
Observations	142	153	152	150	151

These results indicate that a change in the average behaviour of differences and returns occurs. Now Mondays, Tuesdays and Wednesdays are positive, on average, while Thursdays and Fridays are negative. At first sight it seems that the Friday effect is robust through the whole period considered and not a mere transient phenomenon.

2.7.2 Results of Kruskall-Wallis Test

The Kruskall-Wallis Statistic for the differences is 8.09761 with a P-value of 0.0880674.The Kruskall-Wallis Statistic for the returns is 7.7328 with a P-value of 0.101871.This means that, under the assumption of independence, we cannot reject the null hypothesis of equal medians at the 95% of confidence level.

2.7.3 Results of Friedman's Test

The Friedman's statistics for the differences and returns are 5.6800 and 7.8067, respectively. This means that there are not significant differences among the medians in the differences or in the returns.

2.7.4 Regression Models

To complete the series of tests, regression models with dummy variables, heteroscedastic-consistent covariance matrices and ARMA and ARCH effects in the residuals are run for the differences and the returns.

ML-ARCH Dummy Regression. Friday Effect. Differences 2000-2002				
Conditional Mean				
Variable	Coefficient	Std. error	Z-Statistic	Prob.
Mon	0.269429	0.132388	2.035140	0.0418
Tue	0.195062	0.118224	1.649944	0.0990
Wed	0.259474	0.133582	1.942429	0.0521
C	-0.176545	0.070016	-2.521505	0.0117
ARMA Coefficients of the Residuals				
ϕ_1	0.370448	0.163823	2.261272	0.0237
ϕ_5	-0.071144	0.039793	-1.787839	0.0738
ϕ_{12}	0.075136	0.029873	2.515151	0.0119
θ_1	-0.507757	0.157570	-3.222428	0.0013
θ_3	-0.049602	0.039088	-1.268966	0.2045
Variance				
α_0	0.213475	0.125834	1.696477	0.0898
α_1	0.133942	0.054114	2.475187	0.0133
α_2	-0.01958	0.0603	-0.324706	0.7454
α_3	0.304991	0.094057	3.242632	0.0012
α_4	-0.041571	0.130758	-0.317921	0.7505
β_1	0.505665	0.30285	1.669687	0.095
β_2	-0.098994	0.183073	-0.540735	0.5887
β_3	-0.17603	0.134678	-1.307045	0.1912
β_4	0.351998	0.122529	2.872783	0.0041

R-squared	0.029336	**Mean dependent var**	0.011902
Adj. R-squared	0.006354	**S.D dependent var**	1.8232
S.E. of Regression	1.817399	**Akaike info criterion**	3.793562
Sum squared resid	2371.509	**Schwarz Criterion**	3.906093
Log likelihood	-1378.031	**F-statistic**	1.276474
Durbin-Watson stat	1.87449	**Prob (F-statistic)**	0.200559

The best model chosen for the differences has a GARCH(4,4) structure for the residuals. It shows a significant difference between means for Fridays and Mondays at the 95% level and significant difference between Friday and Wednesdays at 90% level. Although this level is hardly ever used, it is included because the related P-value just missed the 95% value.

A similar model for the returns uses the GARCH(3, 3) structure and shows quite similar results. This shows a weakened Friday effect and could mean that the basic causes of the effect are related to market practices related mostly to the bull market period.
The high order autocorrelation in the squared residuals as well as the reduction in the order of autocorrelation of the residuals themselves might be a characteristic signal of the period.

ML-ARCH Dummy Regression. Friday Effect. Returns 2000-2002

Conditional Mean

Variable	**Coefficient**	**Std. error**	**Z-Statistic**	**Prob.**
Mon	0.010884	0.005133	2.120400	0.0340
Tue	0.005174	0.004670	1.108026	0.2679
Wed	0.009555	0.005088	1.877973	0.0604
C	-0.006151	0.002709	-2.270313	0.0232
	ARMA Coefficients of the Residuals			
ϕ_2	-0.834056	0.098204	-8.493128	0.0000
ϕ_5	-0.180533	0.046652	-3.869817	0.0001
ϕ_7	-0.108679	0.036263	-2.996996	0.0027
ϕ_{10}	-0.022186	0.024884	-0.891594	0.3726
θ_1	-0.104936	0.036843	-2.848224	0.0044
θ_2	0.802830	0.099826	8.042328	0.0000
θ_3	-0.190773	0.048165	-3.960806	0.0001

Variance				
α_0	0.000597	0.000228	2.617569	0.0089
α_1	0.075660	0.044351	1.705954	0.0880
α_2	0.080742	0.039461	2.046098	0.0407
α_3	0.191479	0.066595	2.875295	0.0040
β_1	-0.335988	0.200749	-1.673674	0.0942
β_2	0.451131	0.095976	4.700472	0.0000
β_3	0.363208	0.179656	2.021684	0.0432
R-squared	0.038063	**Mean dependent var**		0.000569
Adj. R-squared	0.01535	**S.D dependent var**		0.058238
S.E. of Regression	0.057789	**Akaike info criterion**		-2.927135
Sum squared resid	2.404511	**Schwarz Criterion**		-2.814843
Log likelihood	1098.113	**F-statistic**		1.675859
Durbin-Watson stat	1.939056	**Prob (F-statistic)**		0.042323

Chapter 3

The Search for Possible Explanations

After having discovered a Friday effect, that appears strong during most of the considered period in the differences and returns, a second step is to look for possible explanations. If it is not possible to establish a unique cause for the effect, then the goal should be to reduce the set of plausible reasons one could think of in a first attempt.

3.1 Third Fridays

Since options on whose values the VIX is calculated mature on the Saturday following the third Friday of each month, one should expect prices falling the third week and especially on the third Friday. This could cause differences and returns to be negative on Fridays.
If this is the (only) cause the mean and median of these third Fridays will be negative and large in absolute value, while on the other hand, the median and mean of the rest of Fridays will be quite similar to those of the other days.
However a look to figure 3.1 shows that this is not the case, the statistical tests will have the final word though.

During the period 1988- 2002 there are 177 third Fridays. Their mean and median are shown in the following table together with those of the other Fridays and of the whole period.

Means and Medians of the Third Fridays and Other Fridays

	Differences		Returns		
	Mean	**Median**	**Mean**	**Median**	**Observ.**
Third Friday	-0.272147	-0.340000	-0.005300	-0.009431	177
Other Fridays	-0.200794	-0.230000	-0.004328	-0.005395	579
Total Fridays	-0.219500	-0.260070	-0.010560	-0.014481	756

At first sight, it seems that there are no significant differences between the mean and median of third Fridays and those of the rest of Fridays.

3.1.1 Tests for the Differences

t-tests for Two Population Means

Next t-tests for two population means are run under assumption of equal variances, and then dropping this assumption.

Tests for Equal Means

Differences

	Ass. Equal Variances		Not Ass. Equal Variances	
Altern. Hypothesis	**t**	**P-value**	**t**	**P-value**
Mean1 $\neq$ Mean2	-0.503636	0.614665	-0.531588	0.595381
Mean1$>$ Mean2	-0.503636	0.692668	-0.531588	0.702309
Mean1 $<$ Mean2	-0.503636	0.307332	-0.531588	0.297691

These results show that, under normality assumptions, there are no differences among the means at the 95% confidence level. Since normality cannot be assumed here, the above conclusion should be taken cautiously and it is suggested to try nonparametric tests.

Non-parametric Tests

Mann-Whitney (Wilcoxon) W tests to compare two medians are run:

Tests for Equal Medians

Differences			
	Alternative Hypotheses P-Values		
W	**Median1 $\neq$ Median2**	**Median1> Median2**	**Median1 < Median2**
53817	0.311183	0.844409	0.155591

The results show no statistically significant differences between the two medians at the 95% confidence level.

To have a more sound assessment regarding any differences between the two classes of Fridays the nonparametric **Kolmogorov-Smirnov** test to compare distributions is run.
It gives an estimated overall statistic DN of 0.0671526 and a two-sided large sample K-S statistic of 0.781858. This value has an associated P-value of 0.573922. Therefore the null hypothesis of equal distributions cannot be rejected.

3.1.2 Tests for the Returns

t-tests for Two Population Means

The same t-tests for two population means are run for the differences. The test is run first, under assumption of equal variances, and then eliminating this assumption.

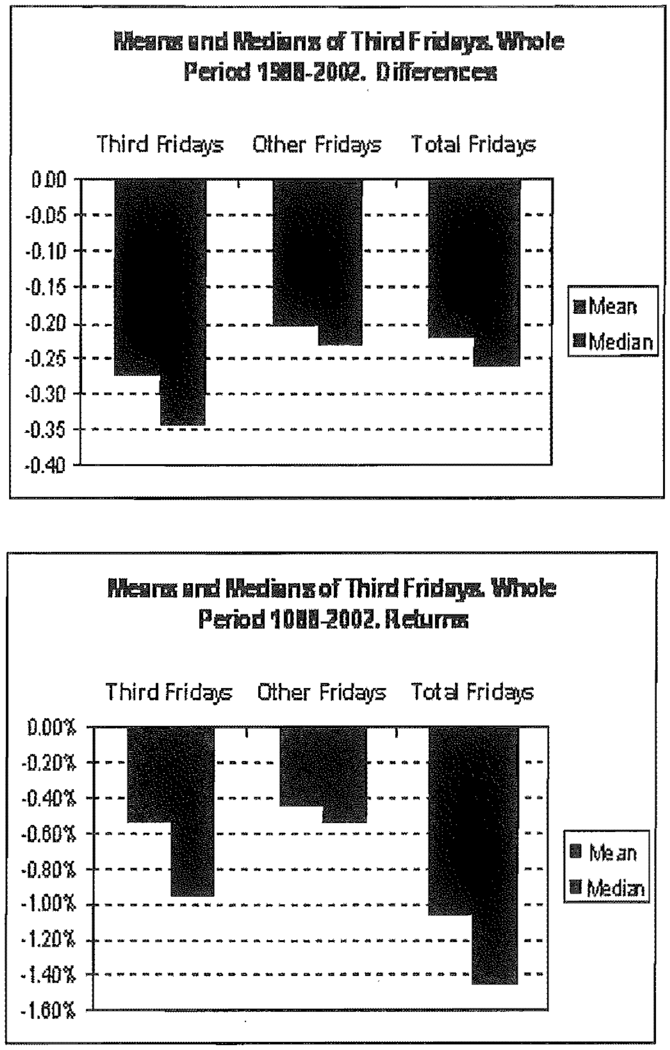


Figure 3.1: Third Fridays and Other Fridays

Tests for Equal Means

Returns

	Ass. Equal Variances		**Not Ass. Equal Variances**	
Altern. Hypothesis	t	P-value	t	P-value
Mean1 $\neq$ Mean2	-0.378952	0.70483	-0.369701	0.711884
Mean1$>$ Mean2	-0.378952	0.647585	-0.369701	0.644058
Mean1 $<$ Mean2	-0.378952	0.352415	-0.369701	0.355942

These results show that, under normality assumptions, there are no differences among the means at the 95% confidence level. Since normality cannot be assumed in the returns, the above conclusion should not be taken literally and, suggests that a nonparametric test should be run.

Non-parametric Tests

Mann-Whitney (Wilcoxon) W tests to compare two medians are run:

Tests for Equal Medians

Differences

	Alternative Hypotheses P-Values		
W	**Median1 $\neq$ Median2**	**Median1$>$ Median2**	**Median1 $<$ Median2**
53340.5	0.409184	0.795408	0.204592

The results show no statistically significant differences between the two medians at the 95% confidence level.

To have a more sound assessment regarding any differences between the two classes of Fridays, the nonparametric **Kolmogorov-Smirnov** test to compare distributions is run.
It yields an estimated overall statistic DN of 0.0906589 and a two-sided large sample K-S statistic of 1.05554. This has an associated P-value of 0.215549. Therefore the null hypothesis of equal distributions cannot be rejected.

In conclusion, **there is not sufficient evidence at the 99% level of confidence, to indicate that OEX options maturing near third Friday cause the Friday effect in the differences or the returns.**

3.2 Possible Seasonal Movements in the S&P100 Index (OEX)

Since the VIX is calculated from OEX option prices, one could suspect that weekly seasonality in VIX is caused by a similar behaviour in the S&P100 index. If such seasonality exists then it should have been discovered and reported a long time ago. Such a seasonal behaviour would be as strong and as manifest as that observed in the VIX.

In the widely known literature there is not a single announcement that weekly seasonality has been discovered in the S&P100 index. In spite of this, during the time in which the first part of this work was done, many tests were run on differences and returns of the data set 1988-2002 and none gave a positive signal of weekly seasonality.
Since OEX is based on blue chips and stocks of this kind have very little risk to lose this value, weekly seasonality in the index (that could cause seasonality in the VIX) is quite improbable.

3.3 Friday Effect Because Of Pre-Holiday Fridays

The data from Mondays in the sample under study are fewer than those of other days of the week, because of several Monday holidays. Do the existence of these Monday holidays, in some way, bring about drops in the VIX value on the respective Fridays before and cause the Friday effect?.

3.3.1 No Day-of-the -Week effect in the Set of Pre-Holiday Data

There are 129 pre-holidays in the sample. Their descriptive statistics is shown below:

Descriptive Statistics. Pre-Holidays 1988-2002. Differences

	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	0.2063	0.0364	0.0252	-0.3504	-0.3966
Median	-0.0050	-0.2100	-0.1800	-0.2900	-0.2300
Std. Deviation	0.69559	0.9747	0.9264	1.6330	1.4233
Kurtosis	0.5026	-0.7047	1.0875	3.2209	2.1501
Skewness	0.9900	0.5071	1.2936	-0.8802	-0.5290
Range	2.1400	3.0400	3.2200	8.1500	8.3700
Minimum	-0.6300	-1.3100	-0.9400	-5.4500	-4.9900
Maximum	1.5100	1.7300	2.2800	2.7000	3.3800
Observations	8	11	21	25	64

Although, it seems that Thursday and Fridays are different from the other days, none of the tests run show statistically significant differences among means or medians.

Descriptive Statistics. Pre-Holidays 1988-2002. Returns

	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	0.0085	0.0009	-0.0033	-0.0124	-0.0171
Median	-0.0002	-0.0122	-0.0099	-0.0165	-0.0142
Std. Deviation	0.0332	0.0472	0.0388	0.0676	0.0588
Kurtosis	0.1088	-0.3777	0.1250	1.0007	0.57251
Skewness	0.7193	0.7453	0.6183	0.0384	-0.1314
Range	0.1035	0.1401	0.1511	0.3149	0.2999
Minimum	-0.0353	-0.0504	-0.0752	-0.1842	-0.1734
Maximum	0.0682	0.0897	0.0759	0.1307	0.1265
Observations	8	11	21	25	64

The same tests are run for the returns with the same results: There are no significant differences among the means or medians.

This means that regarding the set of pre-holidays, there is no statistical evidence for weekly seasonality of the type considered so far in this work.

The means of the differences and and the returns on pre-holidays are -0.24465 and -0.01082, respectively. The respective medians -0.21 and -0.01365. The mean and median for all the differences in the period are -0.00435 and -0.02, respectively. The mean and median for all the returns in the period are -0.00013 and -0.00125 respectively. This means that in average, there is a drop in the closing VIX values on each one these days. These drops are such that there is no statistically significant differences among the means or medians of the differences and returns on those pre-holidays.

Means and Medians on Pre-Holidays

	Differences		Returns	
	Mean	Median	Mean	Median
Pre-Hol.	-0.24465	-0.21000	-0.01082	-0.01356
Non-Preh.	0.00415	-0.02000	0.00025	-0.00080
Total	-0.00435	-0.02000	-0.00013	-0.00125

The, non-parametric, Mann-Whitney (Wilcoxon) Test for difference in medians indicates statistically significant differences betwwen the medians of differences and returns of pre-holidays and those of the other days at the 95 % confidence level.

Man-Whitney Test for Equal Medians.Pre-holidays and Other days.

Differences

	Alternative Hypotheses P-Values		
W	Median1 $\neq$ Median2	Median1> Median2	Median1 < Median2
210325.0	0.03976	0.01988	0.98012

Man-Whitney Test for Equal Medians. Pre-holidays and Other days

Returns

	Alternative Hypotheses P-Values		
W	Median1 $\neq$ Median2	Median1> Median2	Median1 < Median2
206813.0	0.027545	0.013773	0.98623

To have better assessment regarding any differences between the two classes of days, the nonparametric **Kolmogorov-Smirnov** test to compare distributions is run for the VIX differences.
It yields an estimated overall statistic DN of 0.139866 and a two-sided large sample K-S statistic of 1.5612. This has an associated P-value of 0.01527. Therefore the null hypothesis of equal distributions should be rejected.

The same test gives the following results for the returns: Estimated overall statistic DN of 0.135638 and a two-sided large sample K-S statistic of 1.50723.This has an associated P-value of 0.02127. Therefore the null hypothesis of equal distributions should be rejected.

These results mean that there is a statistically significant drop, in average, in the value of the VIX on pre-holidays.

3.3.2 All Days except Pre-Holidays (Other days)

One of the ways to check if the VIX drop, on average, on Pre-holiday Fridays is sufficient to account for the Friday effect is to analyze the set of other days. If this drop is the only cause of the effect, eliminating pre-holydays from the data will cause that the rest of the data will exhibit behaviour reduce the Friday effect in both differences and returns.

Descriptive Statistics. Other Days 1988-2002. Differences					
	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	-0.0379	0.020118	0.0649	0.1613	-0.2031
Median	-0.0350	0.04	0.0550	0.0500	-0.2600
Std. Deviation	1.5720	1.379799	1.2512	1.4661	1.6715
Kurtosis	13.0663	8.772756	3.7273	5.1017	15.3562
Skewness	0.6061	-0.614027	-0.0689	0.9249	2.0316
Range	22.5900	17.56	11.7800	14.2100	20.8500
Minimum	-8.8200	-9.5	-5.9400	-5.7100	-5.8500
Maximum	13.7700	8.0600	5.8400	8.5000	15.0000
Observations	710	762	750	735	692

The Descriptive statistics seem to confirm what is expected in both differences and returns.

Descriptive Statistics. Other Days 1988-2002. Returns					
	Monday	Tuesday	Wednesday	Thursday	Friday
Mean	-0.001061	0.002263	0.002762	0.006465	-0.009948
Median	-0.001931	0.001735	0.003078	0.003154	-0.014672
Std. Deviation	0.061845	0.055793	0.051087	0.057585	0.069654
Kurtosis	4.989624	3.441005	1.141936	0.955999	10.530552
Skewness	0.518004	-0.078279	-0.160599	0.297870	1.864750
Range	0.676724	0.554831	0.380127	0.435257	0.713236
Minimum	-0.254222	-0.322964	-0.195802	-0.190560	-0.192684
Maximum	0.422501	0.231867	0.184325	0.244697	0.520552
Observations	710	762	750	735	692

Results of Kruskal-Wallis Test

The statistic from the Kruskal-Wallis test applied to the differences is 51.9807, corresponding to a P-Value of 1.39231E-10. For the returns the statistic is

55.2567, corresponding to a P-Value of 2.87049E-11.

This means that, under the hypothesis of independence, there are significant differences among the medians at the 99% level.

Results of the Friedman's Test

The Friedman's statistic for the differences is 63.6271 and for the returns 47.5975. Recall that the critical values for a Chi-squared distribution with 5-1 = 4 degrees of freedom at respective levels of 95.0% and 99.0% are 12.592 and 16.812. Then the Friedman's test signals significant discrepancies among the medians in both differences and returns.

Results of the Dunn's Test

T_{ij} Dunn's Stats. Day of the Week Effect. Differences. Non-Prehol

	Monday	Tuesday	Wednesday	Thursday	Friday
Monday	0.0000	1.2805	2.2628	1.6313	4.7361
Tuesday	1.2805	0.0000	0.9823	0.3508	6.0166
Wednesday	2.2628	0.9823	0.0000	0.6315	6.9989
Thursdays	1.6313	0.3508	0.6315	0.0000	6.3674
Friday	4.7361	6.0166	6.9989	6.3674	0.0000

In addition to confirming the Friday effect at 99% level, these results shows significant differences between the medians of Mondays and medians of Wednesdays at 95% level.

T_{ij} Dunn's Stats. Day of the Week Effect. Returns. Non-Prehol					
	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
Monday	0.0000	3.3153	2.7890	2.8417	2.3681
Tuesday	3.3153	0.0000	0.5262	0.4736	5.6833
Wednesday	2.7890	0.5262	0.0000	0.0526	5.1571
Thursdays	2.8417	0.4736	0.0526	0.0000	5.2097
Friday	2.3681	5.6833	5.1571	5.2097	0.0000

This table shows not only the known significant differences regarding Friday's medians but also differences between the Monday's medians and those of the other days at a 99% level.

Regression Models

A regression model for the differences, with ARMA and ARCH effects for the residuals is run. It confirms the Friday effect. No model with ARMA and GARCH effects confirms any effect regarding another days.

ML-ARCH Regression. Friday Effect Non-Preholidays. Diffs 88-02				
Conditional Mean				
Variable	**Coefficient**	**Std. error**	**Z-Statistic**	**Prob.**
Mon	0.250469	0.071487	3.503710	0.0005
Tue	0.242679	0.089733	2.704446	0.0068
Wed	0.228768	0.088760	2.577378	0.0100
Thu	0.291962	0.088900	3.284170	0.0010
C	-0.208873	0.059933	-3.485091	0.0005

ARMA Coefficients of the Residuals

ϕ_1	0.684612	0.110653	6.186993	0.0000
ϕ_3	0.162515	0.086999	1.868015	0.0618
ϕ_8	0.059455	0.017283	-3.440196	0.0006
ϕ_{10}	0.039594	0.022635	1.749218	0.0803
ϕ_{11}	-0.030234	0.022420	-1.348486	0.1775
ϕ_{12}	0.045357	0.019839	2.286181	0.0222
ϕ_{15}	-0.065920	0.020254	-3.254606	0.0011
ϕ_{16}	0.048452	0.019690	2.460745	0.0139
ϕ_{20}	-0.031533	0.014649	-2.152611	0.0313
ϕ_{23}	0.034991	0.015621	2.240034	0.0251
ϕ_{25}	-0.029503	0.015585	-1.893016	0.0584
θ_1	-0.802967	0.106517	-7.538360	0.0000
θ_3	-0.177315	0.100100	-1.771371	0.0765

Variance

α_0	0.111359	0.054723	2.034964	0.0419
α_1	0.173321	0.039491	4.388918	0.000
β_1	0.778892	0.039049	19.946590	0.0000
R-squared	0.050679	**Mean dependent var**		0.003877
Adj. R-squared	0.04541	**S.D dependent var**		1.443683
S.E. of Regression	1.410523	**Akaike info criterion**		3.224017
Sum squared resid	7168.443	**Schwarz Criterion**		3.259917
Log likelihood	-5820.919	**F-statistic**		9.617301
Durbin-Watson stat	2.00246	**Prob (F-statistic)**		0.00000

ML-ARCH Regression. Friday Effect Non-Preholidays. Rets 88-02

Conditional Mean

Variable	Coefficient	Std. error	Z-Statistic	Prob.
Mon	0.011714	0.003693	3.171483	0.0015
Tue	0.014512	0.003434	4.225741	0.0000
Wed	0.015019	0.003354	4.477970	0.0000
Thu	0.017932	0.003430	5.227702	0.0000
C	-0.011878	0.002453	-4.842807	0.0000

ARMA Coefficients of the Residuals

ϕ_1	-0.132439	0.016541	-8.006824	0.0000
ϕ_2	0.777251	0.051545	15.079230	0.0000
ϕ_5	0.133021	0.040306	3.300306	0.0010
ϕ_8	0.058742	0.017980	3.267004	0.0011
ϕ_9	0.030387	0.014516	2.093350	0.0363
ϕ_{10}	0.024672	0.018575	1.328247	0.1841
θ_2	-0.867925	0.047831	-18.145500	0.0000
θ_5	-0.108310	0.045481	-2.381426	0.0172
θ_1	-0.802967	0.106517	-7.538360	0.0000
θ_3	-0.177315	0.100100	-1.771371	0.0765
Variance				
α_0	0.000545	0.000220	2.479357	0.0132
α_1	0.099197	0.026878	3.690616	0.0002
β_1	0.738786	0.068563	10.775260	0.0000

R-squared	0.05097	**Mean dependent var**	0.000273
Adj. R-squared	0.047041	**S.D dependent var**	0.059146
S.E. of Regression	0.057738	**Akaike info criterion**	-2.913574
Sum squared resid	12.07789	**Schwarz Criterion**	-2.886316
Log likelihood	5317.248	**F-statistic**	12.972230
Durbin-Watson stat	2.013366	**Prob (F-statistic)**	0.00000

A similar model is run for the returns. It confirms the Friday effect.
No similar models detect any effect related to other days. The fact that other days show the Friday effect after eliminating pre-holidays, is sufficient to eliminate the possibility of pre-holidays being a cause of such effect.

Chapter 4

The Relationship between VIX and OEX Revisited

4.1 Known Relationships between VIX and OEX

The VIX is usually seen as a proxy for the market's expected volatility over a horizon of 30 calendar days. This has many reasons. For Example:

1. The S&P 100 index (OEX) measures the stock market performance of large U.S. companies. In fact, many fund managers use this index as a benchmark to measure the performance of large capitalization stocks overall.

2. The S&P 100 closely tracks other indices. For instance, in the 1990 decade it had a 0.97 correlation with the S&P 500 and a 0.95 correlation with the DJIA.

3. Feinstein (1989) demonstrates that the implied volatility approximates the market expectation of the average volatility over the life of an option. This approximation is most accurate for at-the-money and near-expiration options.

4. The VIX is calculated from eight OEX options implied volatilities whose

combination represents an at-the-money, thirty-calendar day (22- trading-day) implied volatility.

5. There are documented findings relating daily and weekly VIX changes with OEX level changes (Fleming et al. 1995)

Besides that, the relationships between the S&P 100 implied volatility and the stock market volatility have been largely documented, studied and discussed. [1]

4.2 VIX and OEX Returns over Different Horizons

Since many investors use the returns of the OEX over a fixed horizon (three months, six months, a year, etc.) as a measure of the market's performance, a good starting point is examining the distribution of the returns over each of these periods and then try to establish a relationship of such distribution with the VIX.

4.2.1 Annual Returns

Let us define the annual OEX return as

$$r_t = ln(\frac{P_{t+252}}{P_t}) \tag{4.1}$$

The series so defined over the period 1988 to 2002 behaves nearly as a random walk process [2](see figure 4.1). It is not possible to speak of a distribution of annual returns because of the strong autocorrelations observed:

Although similar to a random walk, this process has a special characteristic: A large partial autocorrelation at lag 251(see fig. 4.2).This phenomenon can be explained as follows: Since P_{t-1} and P_t, consecutive OEX closing values are

[1] See, for instance, Fleming et al. (1995), Christensen and Prabhala (1998) and Fleming (1997)

[2] 252 is the average number of trading days of a year during the period

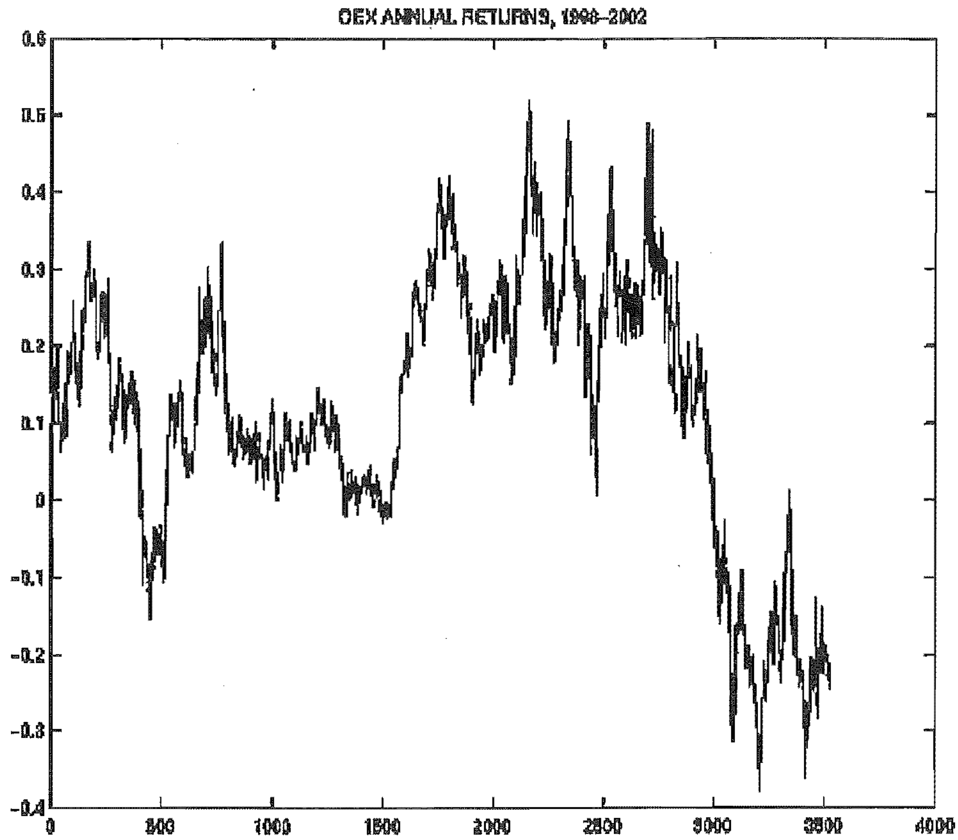


Figure 4.1: OEX Annual Returns. 1988-2002

highly correlated as well as $P_{t-1+252}$ and P_{t+252}, then the respective annual returns are strongly correlated. When the returns correspond to other time periods their series as well as their partial autocorrelation function show similar patterns.

So in the strict sense it is not possible to speak about distribution of returns for these periods. In spite of this, it is possible to build frequency distributions of the OEX returns of certain periods, given the VIX level and to observe very interesting relationships between the statistics related to those distributions and those generated from the OEX returns after the given period.

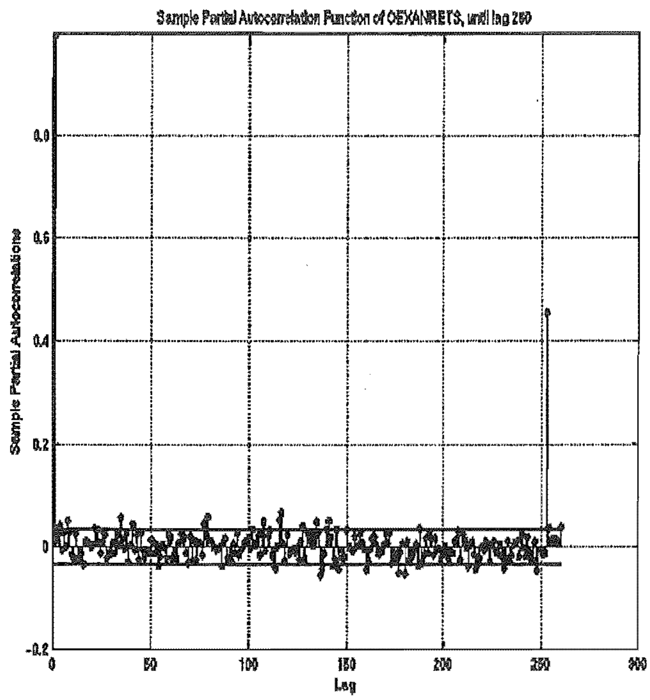


Figure 4.2: PAC of OEX Annual Returns

4.2.2 The General Procedure for the Frequency Distribution

The general procedure for building the frequency distributions given the VIX level is the following: Given a fixed period (three months, for instance), calculate the series of OEX returns for that period. Each return is associated with the VIX level on the day the period starts. The range of the VIX level is divided in intervals of length one and a frequency distribution of OEX returns is created for each interval. Later, the main characteristics of these frequency distributions are studied.

The observed range of the VIX level over the period under study spans from

9.04 to 50.48. So the following intervals are created:
$[9.0, 10.0), [10.0, 11.0) \cdots [50.0, 51.0)$.

Since some VIX values were reached only under extraordinary circumstances, the intervals that contain them might have an "degenerate distribution" (less than two values). For this reason, the interval with the higher value was $[44.0, 45.0)$. The number of trading days for period of three months is 66 and for period of six months, 132.

As an concrete example, let us assume that at time t the VIX closes at 25.60, and the OEX closes at 526.50 and 66 days (three months) later closes at 503.00. Then the associated three-month return is $ln(503.00/526.5) = -0.0457$ this value is taken in account in the frequency distribution associated to the interval $[25.0, 26)$.

4.2.3 The Results from Three- Month Returns

The Means

When the means of the frequency distributions are plotted against the middle point of the respective intervals a curious figure that shows a non-linear relationship emerges, (see figure 4.3). It suggests that, starting at VIX level 27, on average, the more fear that is perceived in the market, the higher the three-month return actually achieved. This seems to be quite related to the trade off between risk and return but, with two differences:

1. The return is not required return but mean return actually achieved
2. This behaviour is not observed at lower levels. At those levels, the mean return alternates between increasing and decreasing.

The Minima

When the minima are plotted Vs. the medium point of each interval, the re-

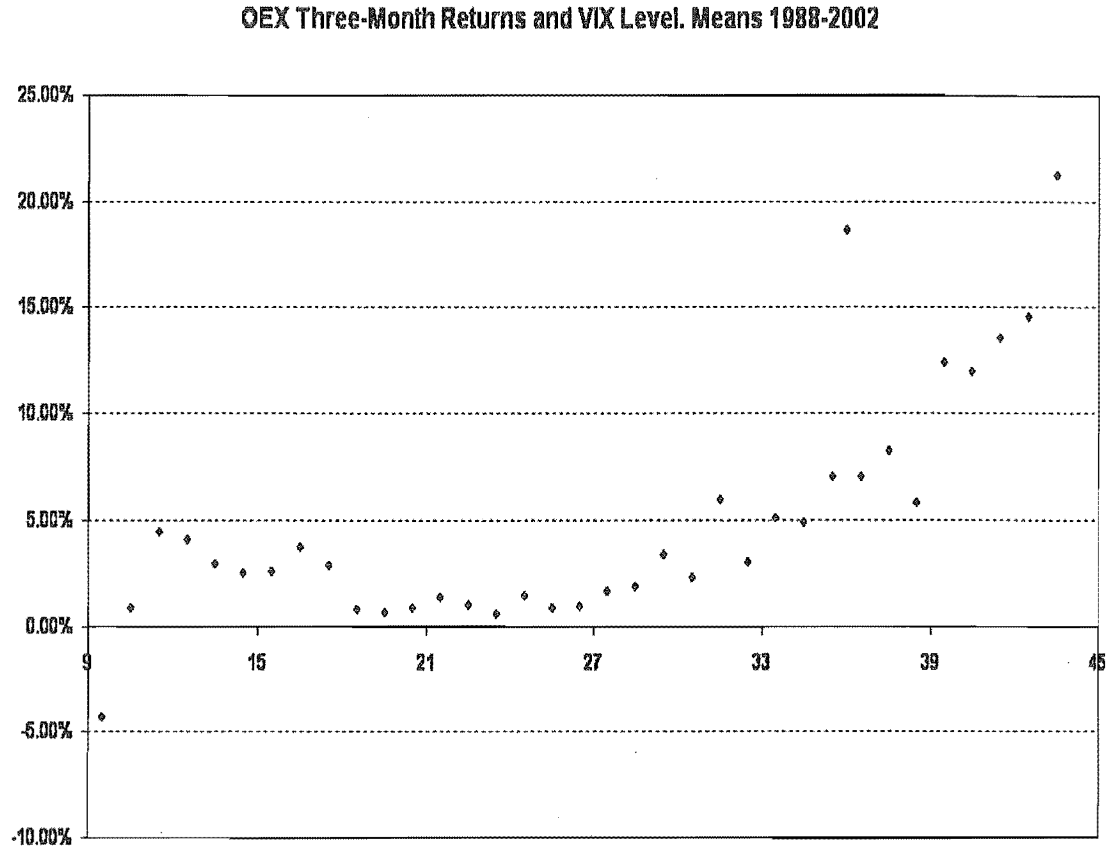


Figure 4.3: Mean OEX Three-Month Returns and VIX Level

lationship shown is even more curious than the previous one (see figure 4.4): Starting from the level 21, the minimum return increases. In particular, for VIX levels of 39 or superior, the minimum obtained return is always positive.

The relationship between minima and the intervals mean values is approximately modelled using the equation [3]

$$Min = 0.0062e^{0.092*Vmean} - 0.03 \tag{4.2}$$

[3]after trying several models this is the best. The fitted model is not shown here

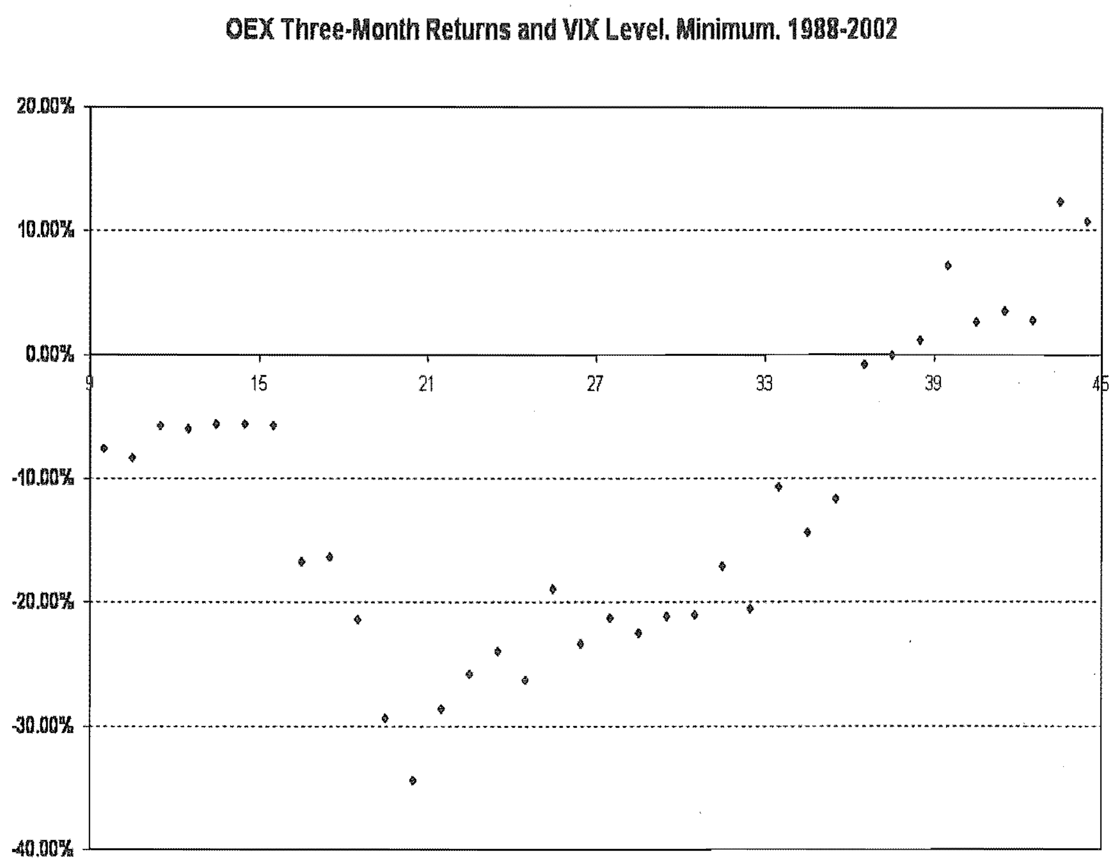


Figure 4.4: OEX Three-Month Returns Minima and VIX Level

where **Vmean** is the midpoint of the interval with the VIX closing values.

When the model is tried with data not in the original sample but in the period November 2002 to June 2003, it gives satisfactory results in the sense that the values predicted by model form a set of lower bounds quite close to the minima actually reached. As a forecast this is better than a naive guess. (See figure 4.5)

Furthermore, the out of sample data show the same behaviour for high values of the VIX. The returns from levels 40 and superior are positive.

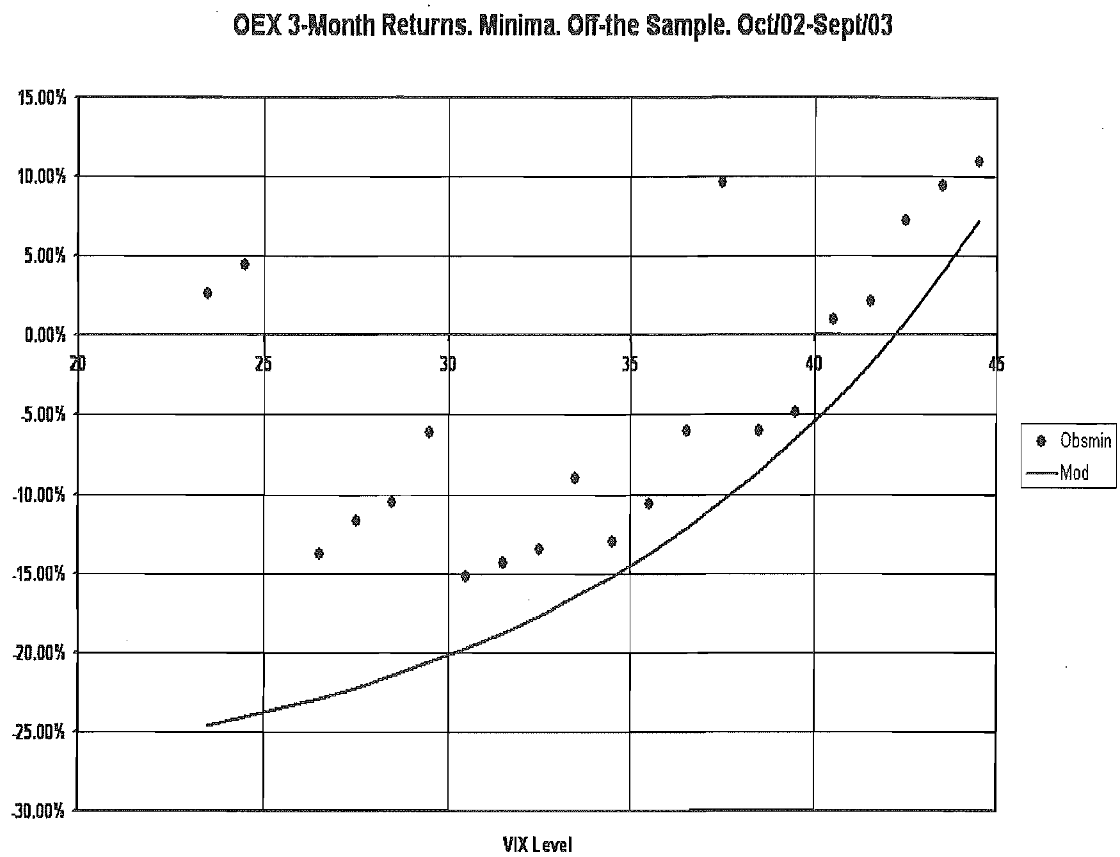


Figure 4.5: Model for Minima and Actual Minima

A similar model is constructed for the means but the dispersion between values predicted and real means is too large to be a satisfactory model.

4.2.4 The Results from Six- Month Returns

The Means

The means graph shows a behaviour similar to that of the respective 3-month returns mean: From level 33 on, the observed OEX mean return increases with the VIX level in a quasi-linear way (see figure 4.6).

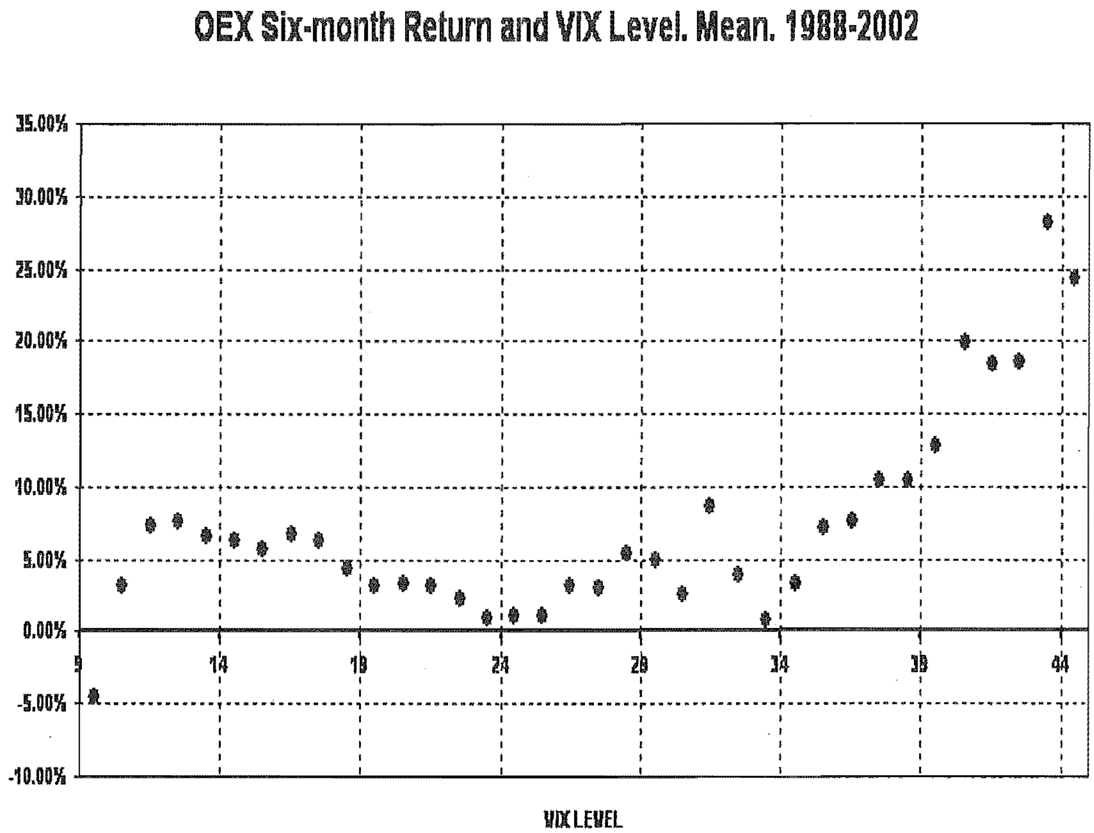


Figure 4.6: Mean Six-Month Returns and VIX Level

The Minima

The minima graph shows behaviour quite similar to that of the minima for the 3-month returns: The minima increase with the VIX level from levels over 21 and, from the level 32 on, the minima are positive.

The following linear model fits the observed minima from 21.5 on :
Min= 0.21*Vmin-0.828.

When applied to the out of sample set of data, it does not act as a lower boundary, as the one related to the 3-month return does, but as a mean minimum around which the observed minimum oscillate. (See figure 4.7)

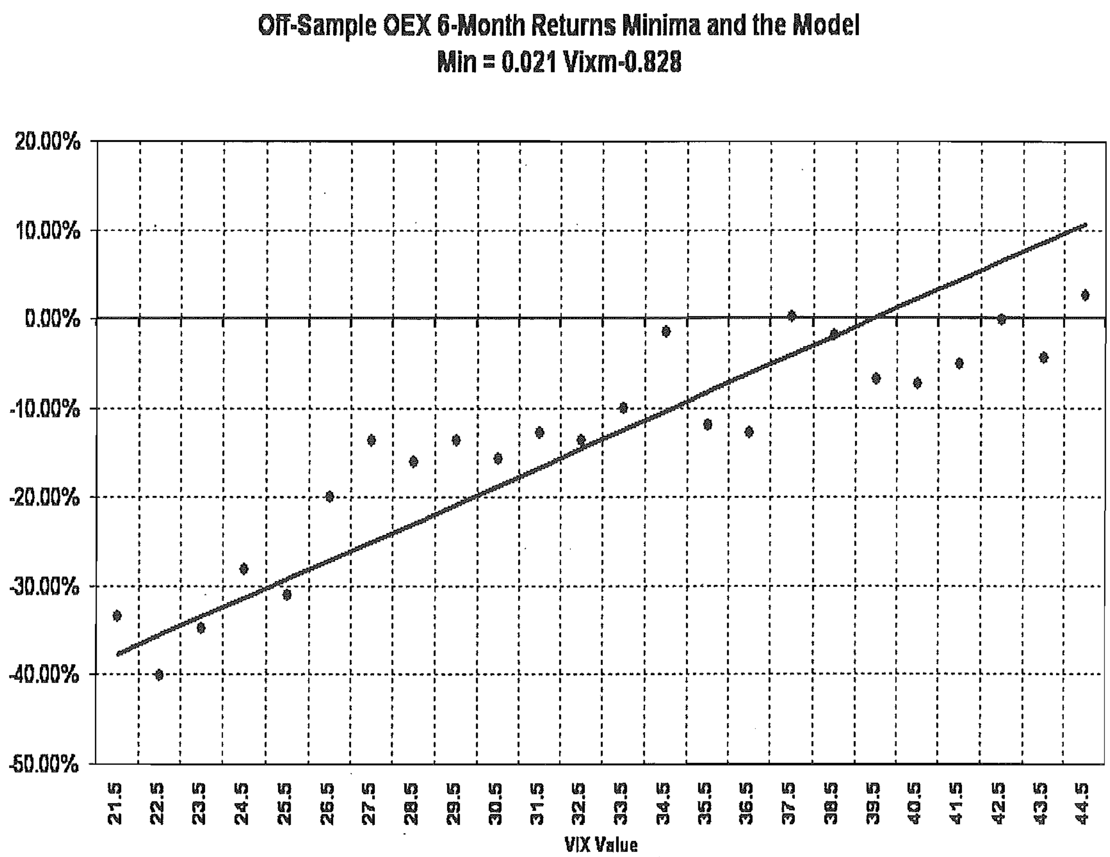


Figure 4.7: Model for Minima and Actual Minima 6-Month Rets.

4.2.5 Results from Annual returns

Given the similar results regarding means and minima from the three and six month returns, one should wonder about if something similar will result from the annual returns. The answer is Yes!. The graphs of means and minima of annual returns look quite alike to those of the three and six month returns. (See figures 4.8 and 4.9)

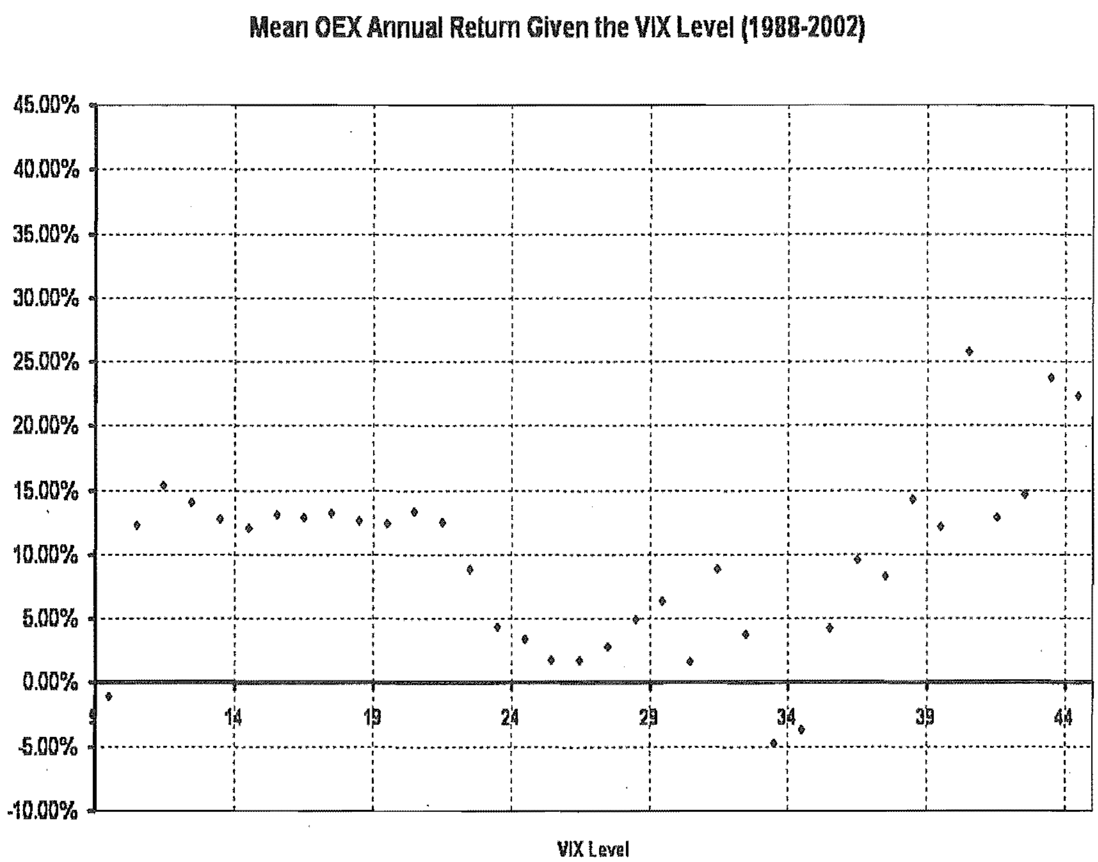


Figure 4.8: OEX Mean Annual Returns Given the VIX Level

Because there are not enough out of sample data points to test them, no models were built for annual returns.

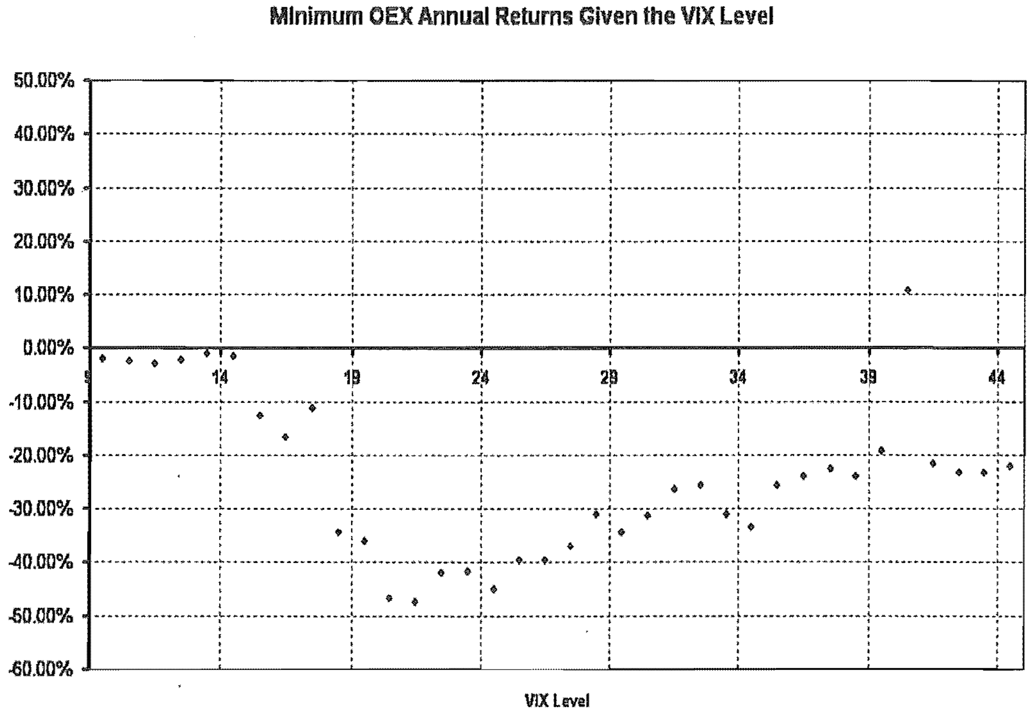


Figure 4.9: OEX Minimum anual Returns Given the VIX Level

4.2.6 The Diminishing Dispersion

When maxima, minima and means are plotted in the same graph, it can be seen that, in general the graph starts with relatively low dispersion, represented by the range (maximum value minus minimum value). For low values of the VIX, the dispersion increases with the VIX level until a value near to 22 or 23 and then, on average, it decreases.
This phenomenon can be observed for all considered horizons. (See figures 4.10, 4.11 and 4.12).

This can be interpreted as a diminution in the variability of OEX returns over the given horizons, when high risk is perceived by investors for the short term. This seems paradoxical because the same phenomenon is observed when there is complacency in the market.

More interesting is the fact that for VIX levels over 39 and horizons of three and six months, the returns are always positive. This fact may cause much theoretical speculation but is better to be cautious and look for more evidence. One possible cause is that fear boosts good market practices related to several horizons and this is reflected, on average, in the OEX. However, this hypothesis needs to be formulated more precisely and tested.

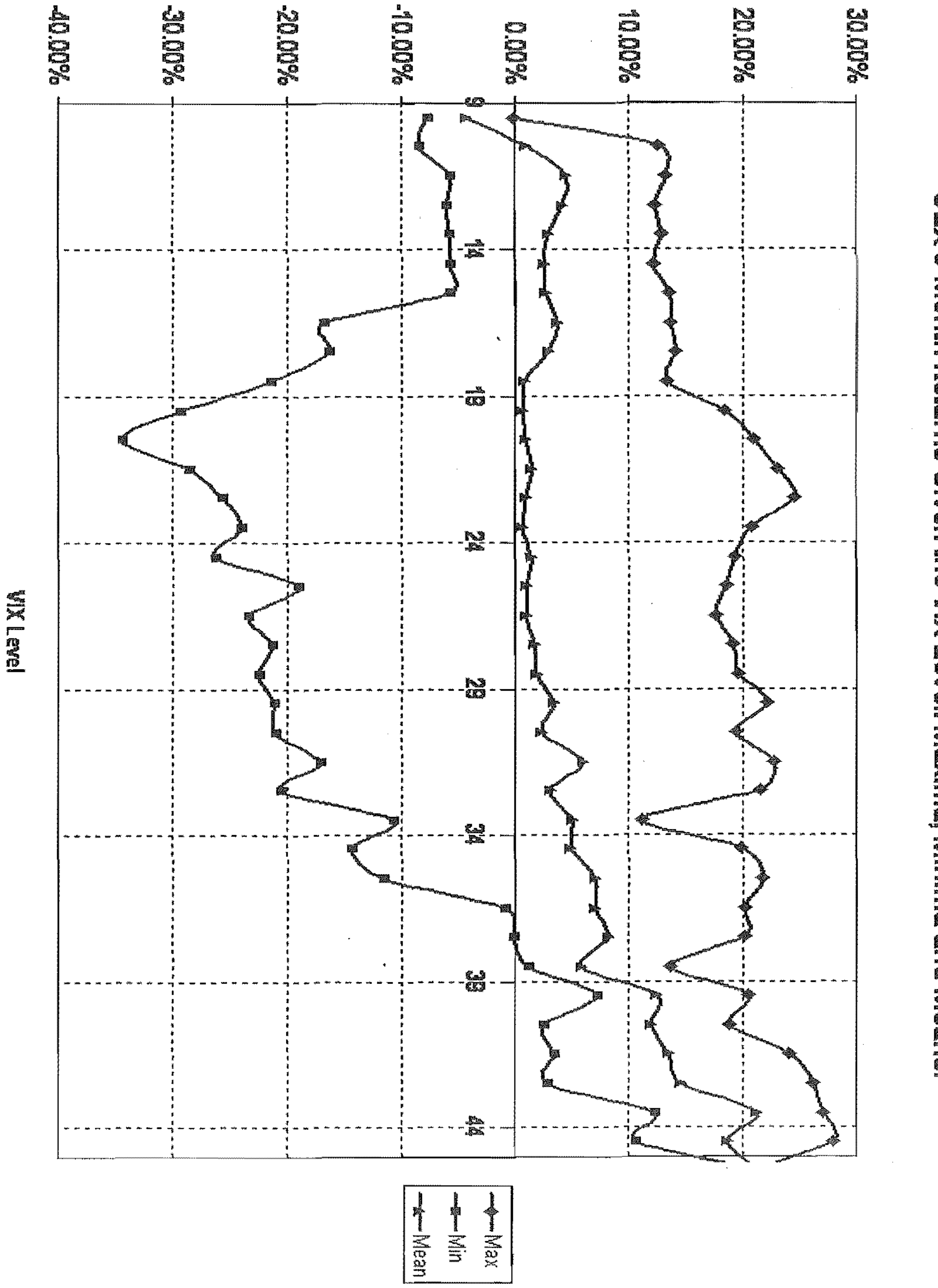


Figure 4.10: Maximum, Minimum and Mean 3-Month Returns

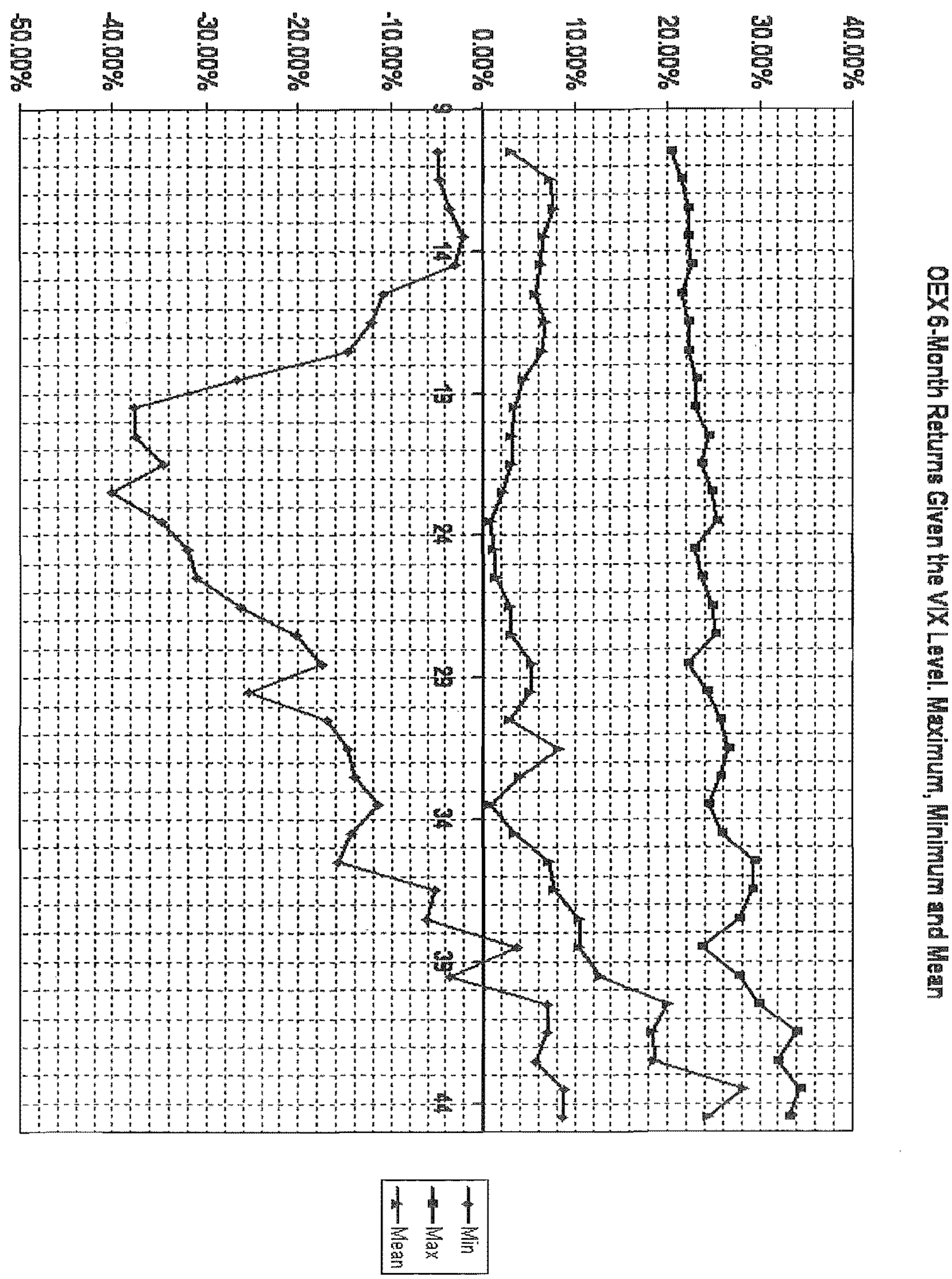


Figure 4.11: Maximum, Minimum and Mean 6-Month Returns

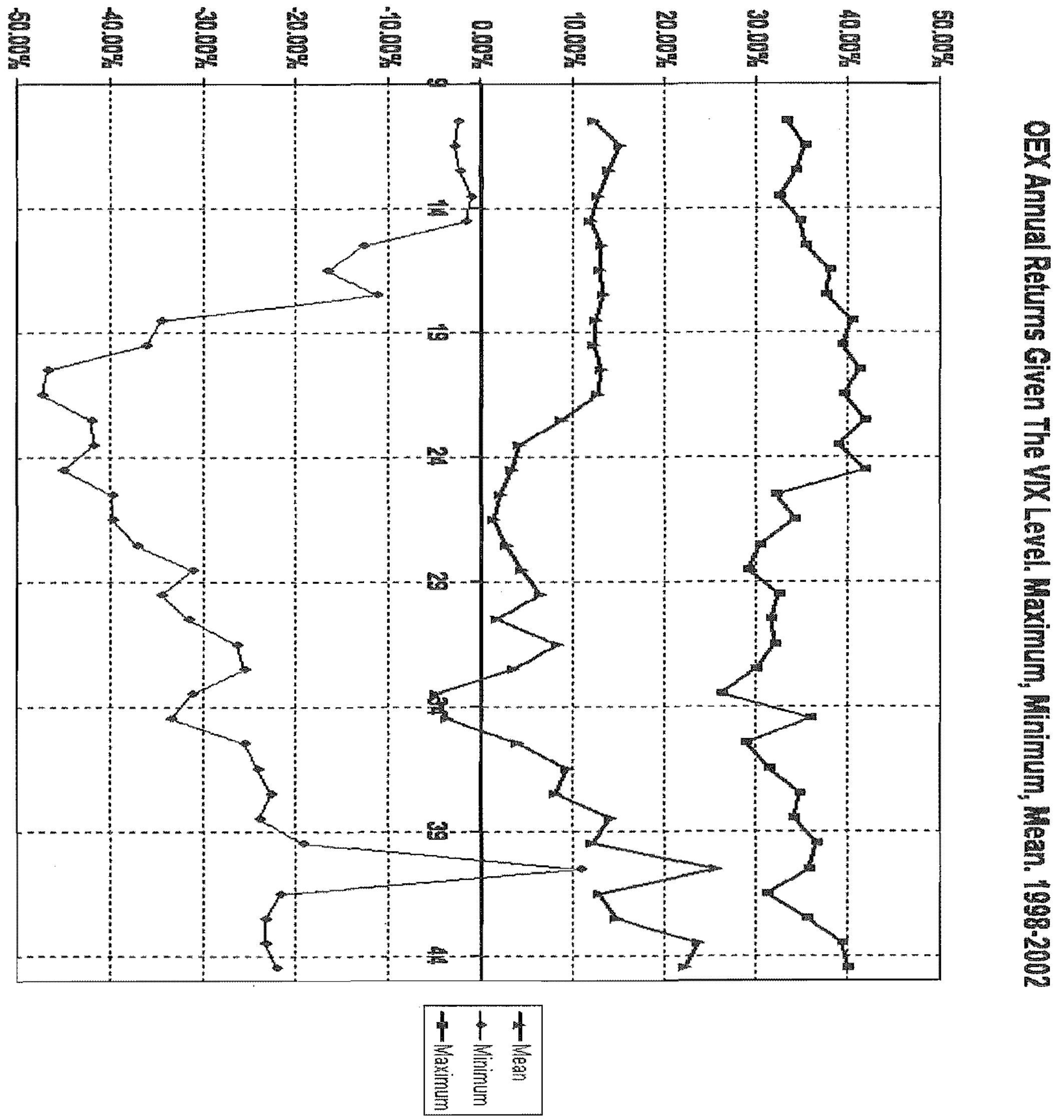


Figure 4.12: Maximum, Minimum and Mean Annual Returns

Chapter 5

The Distribution of the Option's Payoff

In the classical Black-Scholes (B-S) model, the price of the underlying stock is modelled as an Geometric Brownian motion with the two main parameters of drift and volatility.
The payoff at maturity depends on these two factors, since the probability of the stock price to reach values greater than the strike value depend on them. However option prices do not depend at all on the drift, the expected instantaneous (during a short interval of time, dt) rate of return per unit of time of the stock.

Since risk adverse investors need to be offered a risk premium, an appropriate expected return, to take risk, the absence of the drift in the Black-Scholes formula implies that the option's price does not depend on any measure of risk aversion.

The B-S formula is said to be a risk neutral one, since risk neutral people do not need any premium to take risks and such investors make their decisions based on expected values.

Now, the B-S model assumes the existence of a risk free asset (a bond or a bank account) and a tradeable underlying asset. The price of risk free asset evolves related to a deterministic interest rate r. The deduction of the formula needs the creation of a portfolio consisting of the stock and the bond.

In the hypothetical situation in which an option is traded in isolation, no equivalent portfolio can be created,
On the other hand, a speculator who suspects that, for a given horizon, the stock's drift will lead the option's price above of the level given by the B-S model, and wants to make his/her own valuation of the option, will find very useful to know the expected payoff as well as the probability that an option expires without value.
The study of the most important characteristics of the distribution of the option's payoff, within the B-S framework, is the subject of this part of the thesis.

For the speculator and the trader in isolation it is important to know the main features of the distribution of the payoff of calls: The probability of zero payoff, that is, the probability that a call expires out of the money, and the expected payoff. These features are tools to estimate his/her future profits, as long as he/she has a reliable way to estimate the future behaviour of the stock drift.

5.1 Preliminaries

A quick review of the theory supporting the B-S model is necessary at this point to clarify ideas and to establish notation. Here the approach by Björk (2004) is followed.

5.1.1 The Feynman-Kač Lemma

A easy way to solve the B-S equation is using this lemma:

Assume that F is a solution to the boundary problem

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + rF = 0, F(T,x) = \phi(x). \tag{5.1}$$

Let us assume that the process $g = \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)$ is such that $\int_o^t E[g^2(s)]ds < \infty$ and it is adapted to the filtration F_t^W [1]

Then F has a representation

$$F(t, x) = e^{r(T-t)} E_{t,x}[\Phi(X_T)], \tag{5.2}$$

where X satisfies the S.D.E.

$$dX(s) = \mu(s, X(s)).X(s)dt + \sigma(s, X(s)).X(s)dW(s), \quad X(t) = x. \tag{5.3}$$

5.1.2 The Black-Scholes Equation

Assume that the market is specified by the equations:

$$dB(t) = rB(t)dt \tag{5.4}$$

and

$$dS(t) = S(t)\mu(t, S(t))dt + S(t)\sigma(t, S(t))d\overline{W}(t) \tag{5.5}$$

and that we want to price a contingent claim of the form $\aleph = \Phi(S(T))$. Then the only pricing function of the form

$$\Pi(t) = F(t, S(t)), \tag{5.6}$$

which is consistent with the absence of arbitrage must satisfy the following boundary problem in the domain [0, T] X R_+ :

$$\frac{\partial F}{\partial t} + \mu(t, s)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2(t, s)\frac{\partial^2 F}{\partial s^2} - rF = 0 \tag{5.7}$$

$$F(T, s) = \phi(s). \tag{5.8}$$

[1]Intuitively this means that the values of g can be completely determined by observing the trajectories of the Wiener process W for t in the interval [0, T].

By applying the Feynman-Kač Lemma, we know that F has a representation as

$$F(t,s) = e^{r(T-t)}E_{t,s}[\Phi(X(T)], \tag{5.9}$$

where the process X is defined by

$$dX(u) = r.X(u)du + \sigma(u, X(u)).X(u)dW(u), \quad X(t) = s, \tag{5.10}$$

and W is a Wiener process.

Important Remark.
There is a very important difference between equations (5.5) and (5.10)). The first one depends on μ, the expected instantaneous rate of return per unit of time of the stock, while the second one depends on r, the risk free rate.

5.1.3 Martingale Measures

It is customary to speak about two probability measures: The *real* or *objective* probability measure, P, under which the stock price behaves according to (5.5) and the *martingale*, *risk-adjusted* or *risk-neutral* measure, Q, under which the stock prace behaves according to (5.10).

In the first case we speak of the *P-dynamics* , in the second case of its *Q-dynamics.* The expected value in (5.9) is **the expected value of the expression under Q**.

To stress this, the main results of B-S are often presented in the form of the following theorem:

Arbitrage-Free, Simple Claims Valuation Theorem.

The arbitrage-free price of the contingent claim $\Phi(S(T))$ is given by

$\Pi(t;\Phi) = F(t, S(T))$ where F is given by

$$F(t, S(T)) = e^{-r(T-t)} E^{Q}_{t,s}[\Phi(S(T))], \tag{5.11}$$

and the Q-dynamics of S are

$$dS(t) = rS(t) + S(t).\sigma(t, S(t))dW(t). \tag{5.12}$$

5.1.4 The Black-Scholes Formula

For a call option

$$\Phi(S(T)) = Max\{S(T) - K, 0\}, \tag{5.13}$$

and (5.11) can be calculated using

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \quad , \tag{5.14}$$

where

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \tag{5.15}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{5.16}$$

Recall that expression (5.14) is said to be a *risk neutral* formula because it is obtained from (5.11). Hence it does not involve μ but r. It is not related to any risk preference.

5.1.5 Log-Normality Assumptions

If X is a random variable, $X \sim N(\mu_x, \sigma_x)$, and $Y = e^X$ then Y is said to have a log-normal distribution with parameters μ_x and σ_x. It is easily shown that the density function of Y is given by

$$f(y) = \frac{1}{y\sigma_x\sqrt{2\pi}} e^{\frac{(lny-\mu_x)^2}{2\sigma_x^2}}, \quad y > 0. \tag{5.17}$$

The expected value and the variance of Y are given by

$$E[Y] = e^{\mu_x + \frac{1}{2}\sigma_x^2} \tag{5.18}$$

and

$$Var[Y] = e^{2\mu_x}(e^{2\sigma_x^2} - e^{\sigma_x^2}), \tag{5.19}$$

respectively.

The solution to (5.5) is given by

$$S(t) = S_0.e^{(\mu-\frac{1}{2}\sigma^2)(T-t)+\sigma[W(T)-W(t)]} \tag{5.20}$$

and, if t = T we get

$$S(T) = S_0.e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma W(T)}. \tag{5.21}$$

Since the exponent in (5.21) is normally distributed with mean $\mu - \frac{1}{2}\sigma^2$ and standard deviation $\sigma\sqrt{T}$, S(T) is log normally distributed with mean

$$E[S(T)) = S_0 e^{\mu T} \tag{5.22}$$

and variance

$$Var[S(T)) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1). \tag{5.23}$$

5.2 The Distribution of the Payoff for Calls

5.2.1 The Distribution Function

Let us consider a call on the underlying stock with price denoted by S(t), strike price K and maturity T. Its payoff function, Π, is given by

$$\Phi(T) = \Pi(T) = \begin{cases} S(T) - K & if \;\; S(T) \geq K \\ 0 & if \;\; S(T) < K \end{cases} \tag{5.24}$$

The associated distribution function is given by:

$$F_{\Pi_{S_T}}(v) = \begin{cases} 0 & if \;\; v < 0 \\ P(S_T \leq K) = F_{S_T}(K) & if \;\; v = 0 \\ P(S_t > K) = F_{S_T}(K + v) & if \;\; v > 0 \end{cases} \tag{5.25}$$

Here $F_{S_T}(v)$ is the distribution function of the random variable $S_T = S(T)$. It is important to note that the *net payoff* is the result of subtracting the call premium, calculated using the B-S model, from the payoff considered here.

5.2.2 The Density Function

The density function related to (5.25) is given by

$$f_{\Pi_{S_T}}(v) = \begin{cases} 0 & if \;\; v < 0 \\ F(0+).\delta(x) & if \;\; v = 0 \\ f_{S_T}(K + v) & if \;\; v > 0 \end{cases} \tag{5.26}$$

Here F is the distribution function, $\delta(x)$ is the Dirac delta function and $f_{S_T}(v)$ is the density function of the random variable $S_T = S(T)$.

The expressions (5.25) and (5.26) are valid for any model whose underlying asset follows a stochastic process with a probability density at any value of t.

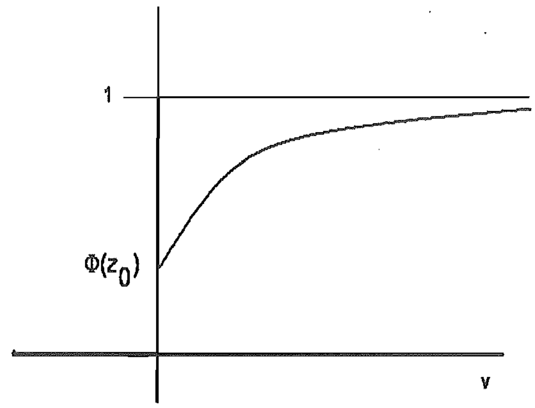


Figure 5.1: Distribution Function of the Call Payoff

5.3 The Distribution and Density Functions in the Black-Scholes Model

A first approximation to the study of the distribution of the payoff of call options can be done within the Black-Scholes framework. This is a first step in understanding the distribution above mentioned in complete markets. [2] Future extensions of this work may include models with non-constant volatility.

Although the Black-Scholes formula does not depend on μ and hence is a risk neutral one, the payoff distribution is quite different. The following analysis is done under the assumptions of this model.

[2]Here the assumptions of the B-S model still hold. If the trader does not hedge, it is because he/she does not want to.

The Probability of Zero Payoff

The study of this probability is quite important since it is the probability that the call option will not be exercised.

Under log-normality assumptions and following (5.26) we have

$$
\begin{aligned}
F_{S_T}(K) = P(S(T) \le K) &= P(S_0 Exp[(\mu - \tfrac{1}{2}\sigma^2)T + \sigma W(T)] \le K] &(5.27)\\
&= P((\mu - \tfrac{1}{2}\sigma^2)T + \sigma W(T) \le ln(\tfrac{K}{S_0})) &(5.28)\\
&= P(W(T) \le \tfrac{ln(\frac{K}{S_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma}) &(5.29)
\end{aligned}
$$

Since $W(T) \sim N(0, \sqrt{T})$, it can be replaced by $\sqrt{T} \in_T, where \in_T \sim N(0,1)$ and the last probability above becomes

$$P(S(T) \le K) = P(\in_T \le \frac{ln(\frac{K}{S_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \tag{5.30}$$

After making

$$z_0 = \frac{ln(\frac{K}{S_0}) + (\frac{1}{2}\sigma^2 - \mu)T}{\sigma\sqrt{T}}, \tag{5.31}$$

the above probability becomes

$$F_{S_T}(K) = P(S(T) \le K) = P(\in_T \le z_0) = \Phi(z_0), \tag{5.32}$$

where this Φ denotes the distribution function of the standard normal.

z_0 is positive if and only if $K > S_0 e^{(\mu - \frac{1}{2}\sigma^2)T}$. Given the nature of the distribution functions, the behaviour or the zero-payoff probability can be studied by studying z_0.

5.3.1 Variations of z_0

z_0, and so the probability of zero payoff, varies with K, S_0, σ, μ and T in the following ways:

Variation of z_0 with K

z_0 increases with K, other things equal:

$$\frac{\partial z_0}{\partial K} = \frac{1}{\sigma K \sqrt{T}} \tag{5.33}$$

This seems to be intuitively obvious. The higher the level K the more diffi-

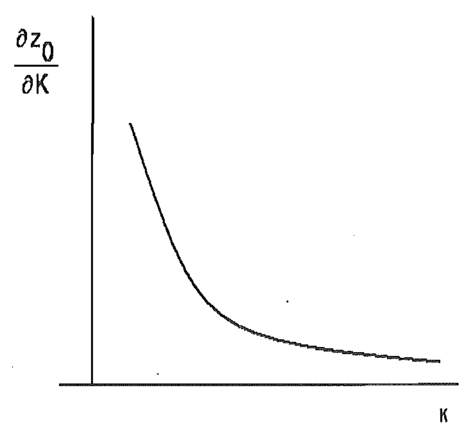


Figure 5.2: Variation of z_0 with K

cult for the stock price to surpass it, given the Gaussian nature of its increments.

Variation of z_0 with S_0

Other things being equal, z_0 diminishes as S_0 increases:

$$\frac{\partial z_0}{\partial S_0} = -\frac{1}{S_0.\sigma\sqrt{T}} \tag{5.34}$$

This is intuitively obvious when $K > S_0$. The higher S_0, the nearer to K and

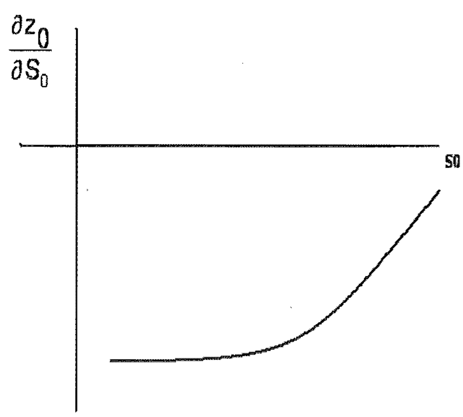


Figure 5.3: Variation of zo with The Inital Value of the Stock

the easier for the price to surpass the K level. On the other hand, if $K \leq S_0$, the level has been already reached from the very beginning and the probability of zero payoff is the probability of the price to descend below the level K.

Variation of z_0 with the Volatility

This is given by

$$\frac{\partial z_0}{\partial \sigma} = \frac{T(2\mu + \sigma^2) - 2ln(\frac{K}{S_0})}{2\sigma^2\sqrt{T}}. \tag{5.35}$$

For fixed values of S_0, K and T, z_0 increases with the volatility in the region in the first two quadrants of the $\mu\sigma$ plane, exterior to the parabola $\mu + \frac{1}{2}\sigma^2 = \frac{1}{T}ln(\frac{K}{S_0})$ and decreases in the interior. The minimum values of z_0 are on that curve (see fig. 5.4).

If $K > S_0$, the vertex of the parabola is in the first quadrant, if $K < S_0$, it is on the second quadrant.

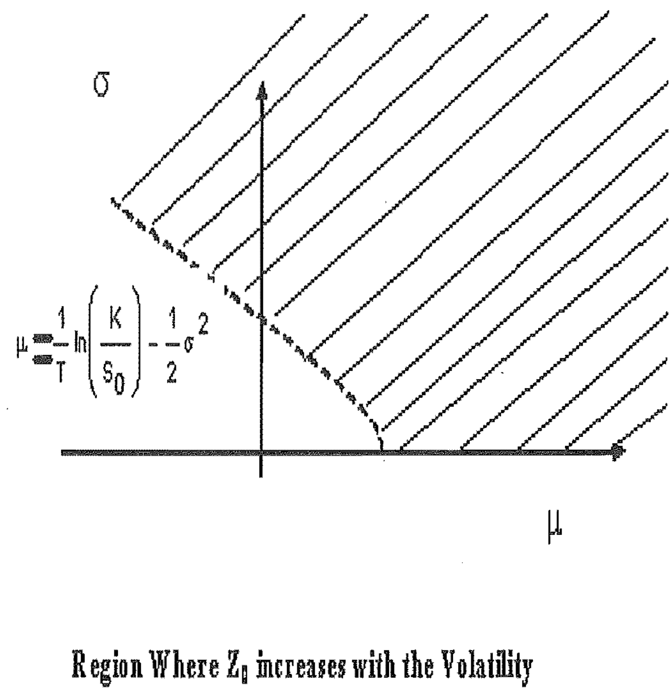


Figure 5.4: z_0 Increasing Region

In general, z_0 is positive if and only if

$$K < S_0 e^{(\mu+\frac{1}{2}\sigma^2)T}. \tag{5.36}$$

The right hand side of (5.33) is the expected value of the log-normal random variable $S(t) = S_0 e^{\mu t + \sigma W(t)}$ for t= T. This stochastic process can be represented by the stochastic differential equation

$$dS = (\mu - \frac{1}{2}\sigma^2) S dt + \sigma S.dW, \quad S(0) = S_0. \tag{5.37}$$

If all parameters except σ are fixed, the minimum value of z_0 is reached on the parabola, for $\sigma^2 = \frac{2}{T} ln(\frac{K}{S_0}) - 2\mu$.

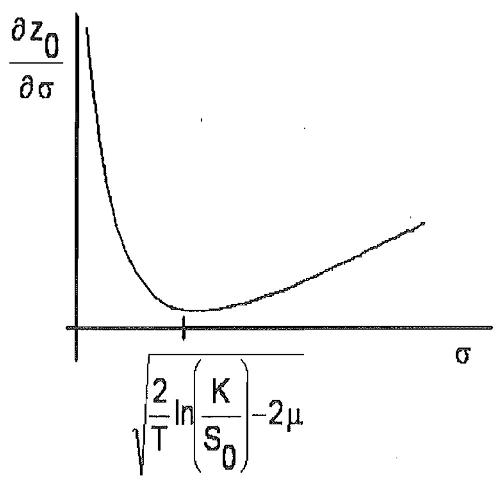


Figure 5.5: Variation of z_0 with the Volatility

Variation of z_0 with the Expected Return

z_0 decreases with μ when other parameters are fixed,

$$\frac{\partial z_0}{\partial \mu} = -\frac{\sqrt{T}}{\sigma} \tag{5.38}$$

Unlike the variation with respect to the volatility, this variation is easily interpreted intuitively. The higher the drift the easier for the price to reach an upper level K and the lower the probability of remaining below that limit.

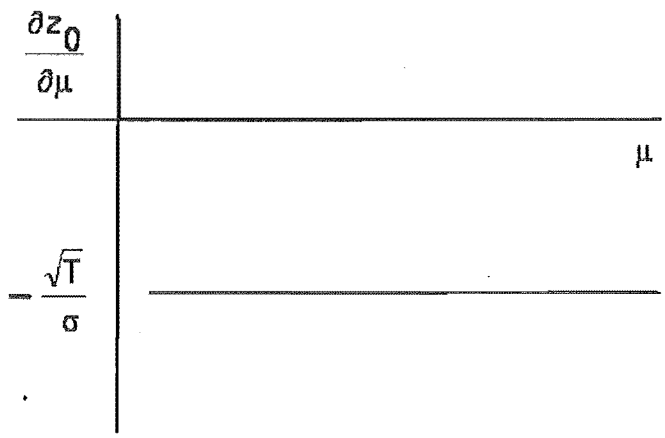


Figure 5.6: Variation of z_0 with the, Instantaneous, Expected Return

Variation of z_0 with the Time to Maturity

This is given by

$$\frac{\partial z_0}{\partial T} = \frac{(\sigma^2 - 2\mu)\sqrt{T} - 2ln(\frac{K}{S_0})}{4\sigma T^{\frac{3}{2}}}. \tag{5.39}$$

This expression will be positive for $\sqrt{T} > \frac{2ln(\frac{K}{S_0})}{\sigma^2 - 2\mu}$ and negative for $\sqrt{T} < \frac{2ln(\frac{K}{S_0})}{\sigma^2 - 2\mu}$, provided that $2\mu < \sigma^2$.

If $2\mu > \sigma^2$, then $\frac{\partial z_0}{\partial T}$ will be positive for $\sqrt{T} < \frac{2ln(\frac{K}{S_0})}{\sigma^2 - 2\mu}$ and negative for $\sqrt{T} > \frac{2ln(\frac{K}{S_0})}{\sigma^2 - 2\mu}$. For most of the observed cases, this last inequality holds.

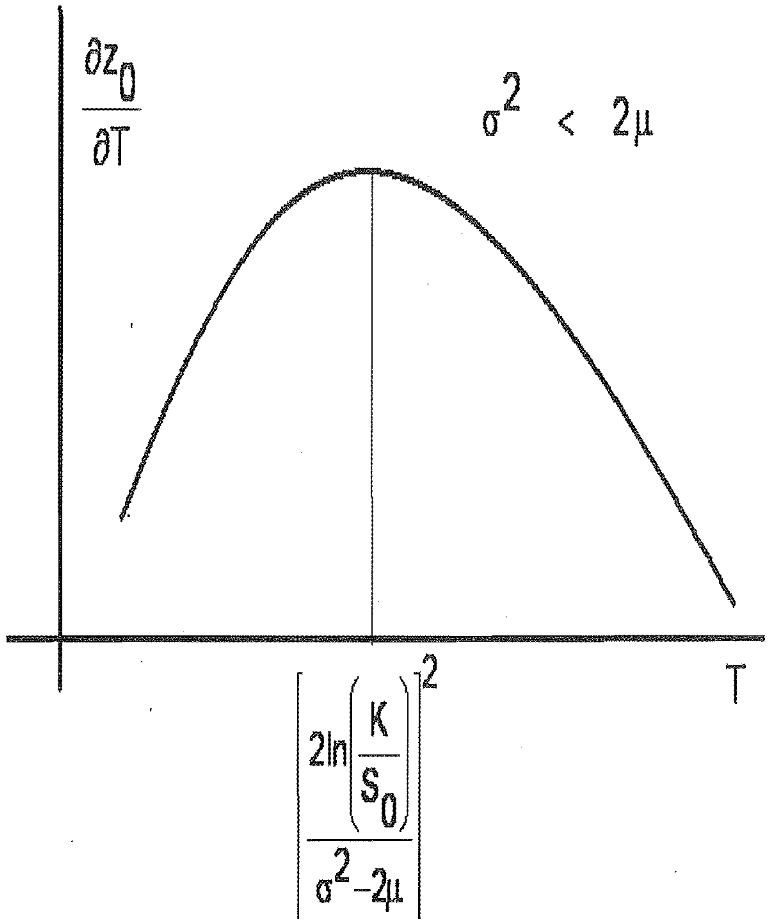


Figure 5.7: Variation of z_0 with the Time to Maturity

Variation of z_0 with the Initial Moneyness $R = \frac{K}{S_0}$

Expressed as a function of the initial moneyness, z_0 becomes

$$z_0 = \frac{ln(R) + (\frac{1}{2}\sigma^2 - \mu)T}{\sigma\sqrt{T}} \tag{5.40}$$

and

$$\frac{\partial z_0}{\partial R} = \frac{1}{R\sigma\sqrt{T}}. \tag{5.41}$$

This expression is always positive

When $R > 1$, the meaning is quite obvious: The higher R, the greater the value of K relative to S_0 also the smaller the value of S_0 relative to K, the more difficult to reach the K level starting from the S_0 level.

The Distribution $F_{S_T}(K+v)$

The second part of (5.25) is expressed in terms of $F_{S_T}(K+v)$ for $v > 0$.

$$\begin{aligned}
F_{S_T}(K+v) = P(S_T \leq K+v) & \\
&= P(S_0.e^{(\mu-\frac{1}{2})T+\sigma W(T)} \leq K+v) \\
&= P((\mu - \tfrac{1}{2}\sigma^2)T + \sigma W(T) \leq ln(\tfrac{K+v}{S_0})) \\
&= P(W(T) \leq \tfrac{ln(\frac{K+v}{S_0})+(\frac{1}{2}\sigma^2-\mu)T}{\sigma}) \\
&= P(\epsilon_T \leq \tfrac{ln(\frac{K+v}{S_0})+(\frac{1}{2}\sigma^2-\mu)T}{\sigma\sqrt{T}})
\end{aligned}$$

By writing

$$Z_1(v) = \frac{ln(\frac{K+v}{S_0}) + (\frac{1}{2}\sigma^2 - \mu)T}{\sigma\sqrt{T}} \tag{5.42}$$

the distribution function can be expressed as

$$F_{S_T}(v) = \begin{cases} 0 & if \quad v < 0 \\ \Phi(z_0) & if \quad v = 0 \\ \Phi(\frac{ln(\frac{K+v}{S_0})+(\frac{1}{2}\sigma^2-\mu)T}{\sigma\sqrt{T}}) = \Phi(z_1(v)) & if \quad v > 0 \end{cases} \tag{5.43}$$

This distribution is neither of continuous nor of discrete type. This is due to the nature of $\Phi(z_0)$.

5.3.2 The Expected Payoff

In working in a world different from a risk neutral one, the expected payoff of a call plays a fundamental role. For instance, it is well known from the Chebyshev's inequality that intervals distant from the mean have low probability measure, irrespective of the type of distribution provided that it has finite first and second order moments.

An Analytic Expression for the Expected Value

If $y = (\mu - \frac{1}{2}\sigma^2)T + \sigma W(T)$, then from (5.24), the expected value of the payoff for the call is given by

$$E(\Pi(T)) = \int\limits_{S_0.e^y \geq K} (S_0 e^y - K) d_{F_y}, \tag{5.44}$$

where d_{F_y} is the density of y.

By using a method similar to that commonly used in deriving the Black-Scholes option pricing formula, a formula for the expected payoff can be obtained.

The deduction is included here for the purpose of future reference and reviewing.

$S_0.e^y \geq K$ if and only if $y \geq ln(\frac{K}{S_0})$, so

$$E(\Pi(T)) = \int_{ln(\frac{K}{S_0})}^{+\infty} (S_0.e^y - K) d_{F_y} \tag{5.45}$$

Now, $y = (\mu - \frac{1}{2}\sigma^2)T + \sigma W(T) = M_T + \sigma W(T)$. Since, $W(T) \sim N(0, \sqrt{T})$,

$$y. \sim N(M_T, \sigma\sqrt{T}) \text{ and } d_{F_y} = \frac{1}{\sqrt{2.\pi T}\sigma} e^{-\frac{(y-M_t)^2}{2T\sigma^2}} dy,$$

$$E(\Pi(T)) = \ = \frac{S_0}{\sqrt{2\pi T}\sigma} \int_{ln(\frac{K}{S_0})}^{+\infty} e^{y - \frac{(y-M_T)^2}{2\sigma^2 T}} dy - \frac{K}{\sqrt{2\pi T}\sigma} \int_{ln(\frac{K}{S_0})}^{+\infty} e^{\frac{(y-M_T)^2}{2\sigma^2 T}} dy$$

$$= \frac{S_0.e^{\mu T}}{\sqrt{2\pi}} \int_{z_0-\sigma\sqrt{T}}^{+\infty} e^{-\frac{u^2}{2}} du - \frac{K.}{\sqrt{2\pi}} \int_{z_0}^{+\infty} e^{-\frac{u^2}{2}} du$$
$$= S_0.e^{\mu T}(1 - \Phi(z_0 - \sigma\sqrt{T})) - K(1 - \Phi(z_0))$$

Now, $1 - \Phi(z_0 - \sigma\sqrt{T}) = \Phi(-z_0 + \sigma\sqrt{T}) = \Phi(\frac{ln(\frac{S_0}{K})+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$, and $1 - \Phi(z_0) = \Phi(\frac{ln(\frac{S_0}{K})+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$

If $D_1 = \frac{ln(\frac{S_0}{K})+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, and $D_2 = D1 - \sigma\sqrt{T}$, then the expected payoff can be expressed as

$$E(\Pi(T)) = S_0.e^{\mu T}.\Phi(D_1) - K\Phi(D_2), \tag{5.46}$$

Where $D_2 = -z_0$.

This result is quite similar to that of the Black-Scholes formula before its present value is calculated. Being the main difference the use of the short rate r instead of μ. That is why D_1 and D_2 were elected for the expressions.

A formula related to (5.44), namely,

$$e^{-\mu T}E(\Pi(T)) = S_0.\Phi(D_1) - e^{-\mu T}K\Phi(D_2), \tag{5.47}$$

that is, the discounted value of (5.44) to t=0, with an interest rate of μ was proposed by Boness (1964) as the fair price of an option under the assumption that "all investors are indifferent to risk."

Variations of the Expected Payoff

This expected value also varies with the same parameters with respect to K, S_0, μ, σ and T.

Variation with K

Since

$$\frac{\partial E(\Pi(T))}{\partial K} = S_0.e^{\mu T}\frac{\partial \Phi(D_1)}{\partial K} - K\frac{\partial \Phi(D_2)}{\partial K} - \Phi(D_2), \tag{5.48}$$

$$\frac{d\Phi(D_1)}{dD_1} = \frac{1}{\sqrt{2\pi}}e^{-\frac{D_1^2}{2}}, \tag{5.49}$$

$$\frac{d\Phi(D_2)}{dD_2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{D_2^2}{2}}, \tag{5.50}$$

$$\frac{\partial D_1}{\partial K} = \frac{\partial D_2}{\partial K} = -\frac{1}{K\sigma\sqrt{T}}, \tag{5.51}$$

and

$$e^{\frac{-D_2^2}{2}} = \frac{S_0}{K}e^{\frac{-D_1^2}{2}+\mu T}, \tag{5.52}$$

then

$$\frac{\partial E(\Pi(T))}{\partial K} = -\Phi(D_2) \tag{5.53}$$

As expected, this derivative is always negative.

Variation with μ

Since

$$\frac{\partial E(\Pi(T))}{\partial \mu} = S_0.e^{\mu T}\frac{\partial \Phi(D_1)}{\partial \mu} + TS_0e^{\mu T}\Phi(D_1) - K\frac{\partial \Phi(D_2)}{\partial \mu}, \tag{5.54}$$

and

$$\frac{\partial D_1}{\partial \mu} = \frac{\partial D_2}{\partial \mu} = \frac{\sqrt{T}}{\sigma}, \tag{5.55}$$

then

$$\frac{\partial E(\Pi(T))}{\partial \mu} = T.S_0.e^{\mu T}\Phi(D_1). \tag{5.56}$$

This means that, other thing being equal, the expected payoff increases with the stock price's drift.

Variation with σ

Since

$$\frac{\partial E(\Pi(T))}{\partial \sigma} = S_0.e^{\mu T}\frac{\partial \Phi(D_1)}{\partial \sigma} - K\frac{\partial \Phi(D_2)}{\partial \sigma}, \tag{5.57}$$

$$\frac{\partial D_1}{\partial \sigma} = -[\frac{ln(\frac{S_0}{K}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma^2\sqrt{T}}] = -\frac{1}{\sigma}(D_1 - \sigma\sqrt{T}), \tag{5.58}$$

and

$$\frac{\partial D_2}{\partial \sigma} = -[\frac{ln(\frac{S_0}{K}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma^2\sqrt{T}}] = -\frac{1}{\sigma}(D_1), \tag{5.59}$$

then

$$\frac{\partial E(\Pi(T))}{\partial \sigma} = \frac{S_0.e^{\mu T}.e^{-\frac{D_1^2}{2}}}{\sigma\sqrt{2\pi}}(D_1 - \sigma\sqrt{T}) + \frac{Ke^{-\frac{D_2^2}{2}}}{\sigma\sqrt{2\pi}}D_1, \tag{5.60}$$

that is,

$$\frac{\partial E(\Pi(T))}{\partial \sigma} = S_0\sqrt{T}e^{\mu T}\phi(D_1), \tag{5.61}$$

where ϕ is the density of the standard Normal distribution.

This means that, unlike the zero-payoff probability, the expected payoff always increases with the volatility.

Variation with the Time to Maturity, T

Since

$$\frac{\partial E(\Pi(T))}{\partial T} = S_o e^{\mu T}\frac{\partial \Phi(D_1)}{\partial T} + \mu S_0 e^{\mu T}\Phi(D_1), -K\frac{\partial \Phi(D_2)}{\partial T}, \tag{5.62}$$

where

$$\frac{\partial \Phi(D_1)}{\partial T} = \frac{\partial \Phi(D_1)}{\partial D_1}.\frac{\partial D_1}{\partial T}; \quad \frac{\partial \Phi(D_2)}{\partial T} = \frac{\partial \Phi(D_2)}{\partial D_2}.\frac{\partial D_2}{\partial T}, \tag{5.63}$$

$$\frac{\partial D_1}{\partial T} = \frac{ln(\frac{K}{S_0}) + (\mu - +\frac{1}{2}\sigma^2)T}{2\sigma T^{\frac{3}{2}}}, \tag{5.64}$$

and,

$$\frac{\partial D_2}{\partial T} = \frac{ln(\frac{K}{S_0}) + (\mu - \frac{1}{2}\sigma^2)T}{2\sigma T^{\frac{3}{2}}} \tag{5.65}$$

then,

$$\frac{\partial E(\Pi(T))}{\partial T} = S_0.e^{\mu T}[\frac{\phi(D_1)}{\sigma\sqrt{T}} + \mu\Phi(D_1)]. \tag{5.66}$$

This last expression is always positive. This means that the expected value increases with the time to maturity.

Variation with the Initial Moneyness

Since

$$\frac{\partial E(\Pi(T))}{\partial R} = S_0 e^{\mu T}\frac{\partial\Phi(D_1)}{\partial D_1}\frac{\partial D_1}{\partial R} - \Phi(D_1)\frac{K}{R^2}e^{\mu T} + K\frac{\partial\Phi(D2)}{\partial D_2}\frac{\partial D_2}{\partial R}, \tag{5.67}$$

where

$$\frac{\partial D_1}{\partial R} = \frac{\partial D_2}{\partial R} = -\frac{1}{R\sigma\sqrt{T}} \tag{5.68}$$

Because of (5.50),

$$\frac{\partial E(\Pi(T))}{\partial R} = -\Phi(D_1)\frac{K}{R^2}e^{\mu T}. \tag{5.69}$$

This expression is always negative, so the expected payoff diminishes with R.

5.3.3 Variance of the Payoff

$$E[\Pi(T]]^2 = S_0^2.\int_{ln(\frac{K}{S_0})}^{+\infty} e^{2y}d_{F_y} - 2.K.S_0.\int_{ln(\frac{K}{S_0})}^{+\infty} e^y d_{F_y} - K^2\int_{ln(\frac{K}{S_0})}^{+\infty} d_{F_y}, \tag{5.70}$$

so

$$E[\Pi(T]]^2 = S_0^2.e^{-m_T}\Phi(D_3) - 2KS_0e^{\mu T}\Phi(D_1) - K^2\Phi(D_2), \tag{5.71}$$

where

$$m_T = T(\frac{\mu^2 + \sigma^2 + \frac{5}{4}\sigma^4}{\sigma^2}), \tag{5.72}$$

and

$$D_3 = \frac{ln(\frac{S_0}{K}) + (\mu + \frac{3}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{5.73}$$

Given the forms of (5.44) and (5.66), there does not seem to exist a simple expression for the variance.

5.4 The Probability of Zero Payoff within the B-S Model for Real-World Data

Before doing any empirical study regarding the probability of zero-payoff, it is important to have estimates of some limits for the most important parameters. That is, using some real world stocks and indexes to get some estimates of the drift and the volatility on an annual basis.

5.4.1 Estimates for the Drift and Volatility

If $x_1, x_2, \ldots, x_n$ are the logarithms of the observations, then an estimator of the mean value, $m = (\mu - \frac{1}{2}\sigma^2)T$, is

$$\hat{m} = \frac{1}{n}(\sum_{k=1}^{n} x_k). \tag{5.74}$$

An estimator of the variance, σ^2, is

$$\hat{\sigma}^2 = \frac{1}{n-1}(\sum_{k=1}^{n}(x_k - \hat{m})^2). \tag{5.75}$$

Therefore, an estimator of μ is

$$\hat{\mu} = \hat{m} + \frac{1}{2}\hat{\sigma}^2, \tag{5.76}$$

provided that T=1.

The following Matlab program was used to calculate some estimates of μ and σ for the OEX index and IBM and Conoco stocks.

```
   %This program calculates annual estimates for mu and sigma
[A1, A2]=textread('Conph8203.txt','%s%f', 'headerlines', 1);
N=length(A1);
Nyears=floor(N/250);
Vecyears=zeros(Nyears,1);
Vecmean=zeros(Nyears,1);
Vecstd=zeros(Nyears,1);
Vecmu=zeros(Nyears,1);
s=1; %Counter of file pointer
s1=char(A1(s));
localv=findstr(s1,'-');%it finds the first position of '-'
localy=localv(2);
syear=strcat(s1(localy+1),s1(localy+2));%string of the year
syear1= syear;
yr=1;
j=1;
while  s < N
  Vecyear=zeros(300,1);
   Vecyears(j)=str2num(syear);
%number of years
k=1;
Vecyear(j,1)= A2(s);
while  (syear1== syear) & (s < N)
   Vecyear(k,1)= A2(s);
  k=k+1;
  s1=char(A1(s));
  localv=findstr(s1,'-');%it localizes the first position of '-'
  localy=localv(2);
```

```
  syear=strcat(s1(localy+1),s1(localy+2));%string of the year
  s= s+1;
end
Vecyear= Vecyear(1:k,1);
Vecmean(j)=mean(Vecyear);
Vecstd(j)=std(Vecyear)
j=j+1;
syear1= syear;
end;
Vecmean=365*Vecmean;
 Vecstd=sqrt(365)*Vecstd;

   Vecmu= Vecmean+ 0.5*Vecstd.^2;
```

The results from this program are shown in table 5.1.

Those results are used as a guide for the study of the probability of zero payoff and the expected payoff. Payoff tables were calculated for values of the drift in the range -0.60 to 0.60 and for values of the volatility in the range 0.1 to 1.0 with increments of 0.05 in both parameters.

For the tables to be useful for any initial and strike values of the stock, they were constructed for values of the ratio $R = \frac{K}{S_0}$ or *initial moneyness.* The ratio varies from 0.95 to 1.15 with increments of 0.01 for each value of the time to maturity T=0.08333(one month), T=0.25 (three months) and T=0.5 (six months).

Although it is not probable that anyone *rational* buys (and that anyone *rational* writes) a call option whose strike price is less than its current value and it is quite unlikely that the stock price rises 15 percent in a month, these ranges help in showing the relationship between the zero-payoff probability and the drift and the volatility. For those *non-rational,* a possible interpretation is given in next section.

Table 5.1: Estimates of Drift and Volatility for OEX and Three Stocks

Year	OEX		IBM		Conoco-phil		Coca-Cola	
	Sigma	Mu	Sigma	Mu	Sigma	Mu	Sigma	Mu
1986	0.19	0.19	0.25	-0.31	0.43	0.19	0.33	-0.99*
1987	0.43	0.17	0.48	0.10	0.63	0.48	0.54	0.20
1988	0.22	0.10	0.26	0.04	0.45	0.61	0.31	0.04
1989	0.17	0.38	0.21	-0.28	0.39	0.54	0.29	0.15
1990	0.20	-0.11	0.24	0.22	0.32	0.11	0.40	0.05
1991	0.18	0.35	0.29	-0.27	0.31	0.04	0.38	0.03
1992	0.12	0.04	0.33	-0.79	0.32	0.17	0.29	-0.27
1993	0.11	0.12	0.36	0.27	0.36	0.36	0.35	0.32
1994	0.12	0.00	0.36	0.42	0.31	0.23	0.33	0.37
1995	0.10	0.47	0.28	0.34	0.25	0.14	0.32	0.67
1996	0.15	0.29	0.40	0.83	0.27	0.45	0.33	0.84
1997	0.23	0.39	0.93	-0.10	0.28	0.21	1.37	0.50
1998	0.25	0.42	0.37	0.86	0.37	-0.04	0.50	0.12
1999	0.23	0.41	1.00	-0.16	0.35	0.18	0.53	-0.65
2000	0.29	-0.20	0.57	-0.29	0.42	0.44	0.76	0.17
2001	0.28	-0.14	0.45	0.63	0.35	0.14	0.48	0.19
2002	0.33	-0.31	0.51	-0.46	0.34	-0.23	0.43	0.29
2003	0.21	0.28	0.28	0.24	0.21	0.40	0.30	0.02

***: very few data for good estimates**

5.5 Interpretability and Usefulnes of the Results

Call options are mainly bought for people who are optimistic about the inmediate future of the price of the underlying stock. If the price exceeds the excercise price, in the ideal market, they will have a net profit by excercising the option and selling the stock.

Now, let us consider the hypothetical situation in which we have two market participants:

A worried owner of a stock who has seen its price going down for several months and needs liquidity, so he will sell the stock for a lower price (not too low though).
A contrarian who is looking for a profit not in the inmediate future but later, and he hopes that, as in many similar situations, after reaching a minimum the price will rise quickly.
Under these circumstances, the worried owner can issue a call option with time to maturity T and strike price $K < S_0$. If the call option expires in the money, the buyer will have an "inmediate profit" from buying the stock at a lower price than the market price as he will hope for a future profit when the price rises again. Also, if in spite that the call expires out of the money, he believes that the price will rise anyway, he can exercise the option. On the other hand, the owner will have some liquidity. A prolonged descent might imply a negative drift (at least temporarily) so some of the graphics we have here could be useful.

For instance, if the current price of the stock is \$24, the drift is estimated in -0.10 and the volatility in 0.20, the owner writes a call option with time to maturity one month and strike price $K = 22.8$ so R=0.95. From figure 5.3, for T = 1 month, the estimate probability of the stock to finish out of the money is 0.25, so the probability that the owner will get liquidity in one month is 0.75.

5.6 Putting the Results Together: 1. Graphs of the Zero-Payoff Probability

Although the tables are very useful for calculations and estimations, they do not give a complete picture of the phenomenon under study. A group of graphs of the probability of zero-payoff for calls was created. Each shows the probability as a function of the volatility and the drift, for a given R.

5.6.1 Results for R= 0.95

Figures on page 114 show how for R=0.95 and T= 0.0833 (1 month), the probability increases with σ in a quasi-linear way, while it decreases almost exponentially with the drift, for low values of the volatility.

This is shown in more detail in figures (5.16) to (5.19):
For low values of the volatility, the probability diminishes almost exponentionally with the drift, but as the volatility increases, around 0.5, the diminution becomes almost linear.

For any value of μ, the zero-payoff probability increases with the volatility. The lower the drift value, the higher the probability, rising above 0.55 for $\sigma = 1$. Very low values of μ lead to an increase almost linear, higher values to an increasing similar to the logaritmic one.

Figures (5.20) and (5.21 show the behaviour of the zero-payoff probability for R=1.0 and T= 0.8333.

This is detailed in figures (5.22) to (5.25):
The probability always decreases with μ.
For lower values of the volatility, the probability decreases with μ in a way similar to that of a logistic function reflected on the line $y = 1$, as the volatility increases the curve straightens, when σ reaches 0.55 the curve starts to warp again in a way such that for negative values of the drift, higher values of the volatility give lower values of the probability and vice versa. But, for higher values of μ, the lower the volatility, the lower the probability.

As it is shown in figures (5.24) and (5.25), for negative values of μ the probability decreases with σ and, for zero and positive values of μ the probability increases with sigma, however, for volatility values over 0.6, it increases or decreases so slowly that it seems almost constant.

5.6.2 Results for R=1.05

For R= 1.05, the probability of zero payoff almost always decreases with both drift and volatility.Figures (5.26) and (5.27) give the global picture from two different points of view.

Figures (5.28) and (5.29) show how the decrease with the drift changes. For low values of the volatility it is variable and negative, but as the volatility increases, it becomes almost constant until the value 0.60 when the curves start to bend again but slowly. The behaviour is such that for negative values of μ, the lower the volatility, the higher the probability and this continues until $\mu = 0.25$, approximately, when the behaviour invertes. The lower the volatility, the lower the probability.

Figures (5.30) and (5.31) shows how, except for values of the drift over 0.50, the probability decreases with σ. For values of μ over 0.50, the probability increases slightly.In general, for values of σ over 0.7, the variations in the probability are so small that practically there is no change with the volatility.

5.6.3 Results for R=1.10

Figures (5.32) and (5.33) show how the probability of zero payoff diminishes with both μ and σ. This is shown in more detail in figures (5.34) to (5.37).
In figures (5.34) and (5.35) it is clear that the lower the volatility the higher the probability for any given value of the drift. Figures (5.36) and (5.37) show how, for any given value of σ, the higher the drift, the lower the probability.These graphs also show that even for high values of the drift and the volatility, for that value of R, the probability of zero-payoff remains above 0.60.

5.6.4 Results for R=1.15

The general characteristics of how the zero-payoff probability varies with σ and μ are shown in figures (5.38) and (5.39).

Figures (5.40) and (5.41) show how the probability diminishes with the volatility for every given drift value. The higher the volatility, the lower the probability.

Figures (5.42) and (5.43) show how the probability diminishes with the drift, for every given volatility level: The higher the drift, the lower the probability.

5.7 Putting the Results Together: 2. Graphs of the Expected Payoff

To get some results useful for any initial price of the underlying stock, expected payoffs are not calculated but a factor multiplied by S_0 can give the expected payoff. Each factor can be interpreted as a percentage of the original price of the stock, S_0.

Equation (5.44) can be written as

$$E(\Pi(T)) = S_0.F, \tag{5.77}$$

where,

$$F = e^{\mu T}\Phi(D_1) - R\Phi(D_2). \tag{5.78}$$

Thereafter tables of factors can be calculated for each convenient value of R.

Here the correspondent graphs are shown instead of the tables. As in the case of the zero-payoff probability, the tables themselves can be used to estimate some required values of the expected payoff.

Multipliers for R=0.95

Figure (5.44) shows how the expected payoff increases with both the drift and the volatility.

Figures (5.45) and (5.46) show in detail the variation with the drift. It always increases, and for a given value of μ, the higher the volatility, the higher the expected value.

Figures (5.47) and (5.48) show the variation with the volatility with details. It always increases, and for a given value of the volatility, the higher the drift, the higher the expected value.

Multipliers for R=1.00

Figure (5.49) shows how, although the expected value increases with both the volatlity and the drift, the first one has a greater effect.

Figures (5.50) and (5.51) show the effect of the drift with details: even for high values of the drift, usually observed values of the volatility do not produce great expected values.

Figures (5.52) and (5.53) show the effect of the volatility and drift with details: Given a level of volatility, the expected value increases with μ, although the variation is quite low.

Multipliers for R=1.05

Figure (5.54) shows how as R increases, the importance of μ as a determinant of the expected payoff diminishes and how σ becomes the main factor.

Figures (5.55) and (5.56) show the variation of the expected payoff more detailed: Even for a level of the drift as high as 20, and usually observed values of the volatility, the expected value does not rises above the 1.5 percent.

Figures (5.57) and (5.58) show the variation with the volatility and how, given a volatilitylevel, increasing the drift level increases the expected value only slightly.

Multipliers for R=1.10

Figure (5.59) shows how for this value of R, the drift is almost irrelevant and the most important factor determining the expected payoff is the volatility.

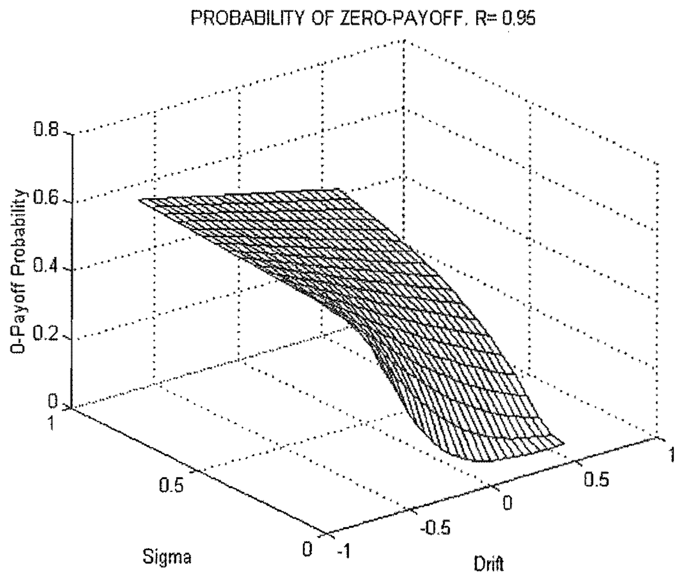
Figures (5.60) and (5.61) show the variation of the expected payoff with the drift with more details. Only for very high levels of volatility there area high expected payoff, for any given level of the drift.

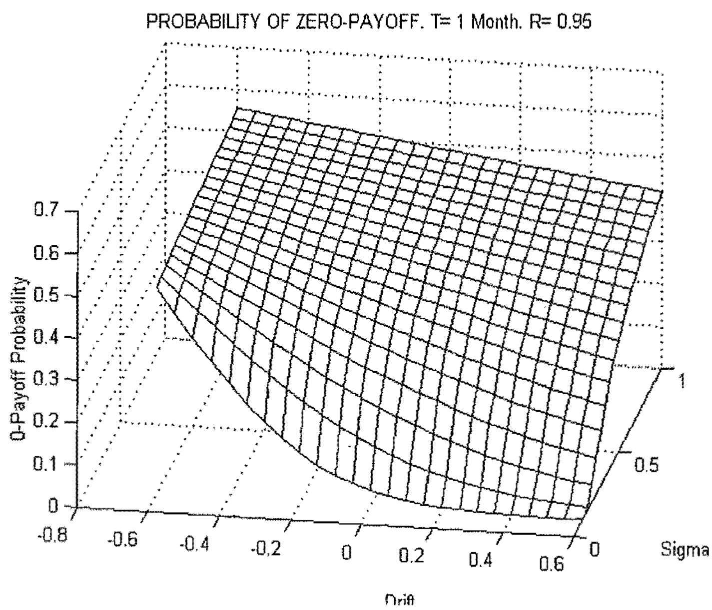
Figures (5.62) and (5.63) show the variation with the volatility. In certain sense it is similar to that of the R=1.05.

Multipliers for R=1.15

Figure (5.64) shows how for this level of R, the expected payoff is almost constant for any drift level and low levels of volatility.
Although, in practice, levels of R such as 1.10 and 1.15, are not used for maturities of 1 month, including them helps in get a general idea of the joint dependence of the payoff level on the volatility and the drift.

Figures (5.65) to (5.68) help to characterize this dependence and to estimate the multipliers.


PROBABILITY OF ZERO-PAYOFF. R= 0.95
0-Payoff Probability
0.8
0.6
0.4
0.2
0
1
0.5
0
Sigma
-1
-0.5
0
0.5
1
Drift


PROBABILITY OF ZERO-PAYOFF. T= 1 Month. R= 0.95
0-Payoff Probability
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
Drift
0
0.5
1
Sigma

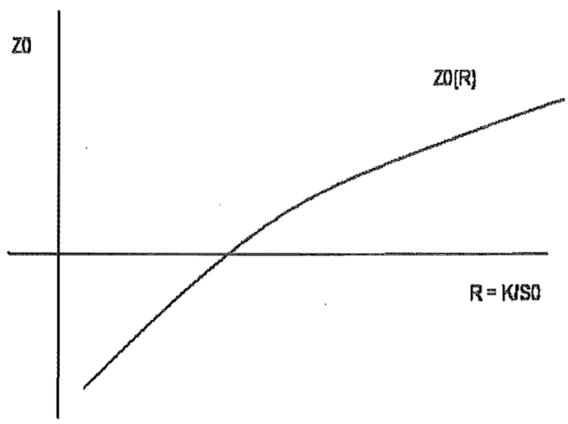


Figure 5.8: z_0 as a Function of the Initial Moneyness

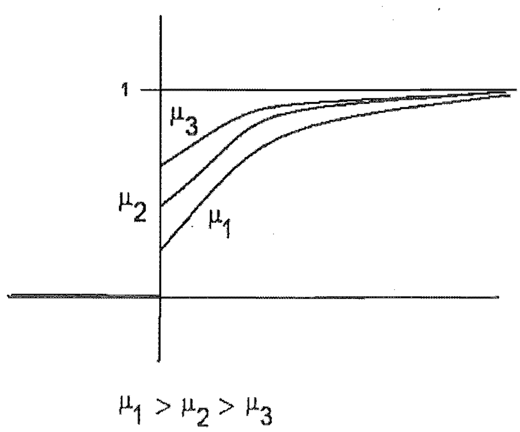


Figure 5.9: Distribution Function of the Payoff for Different Values of μ

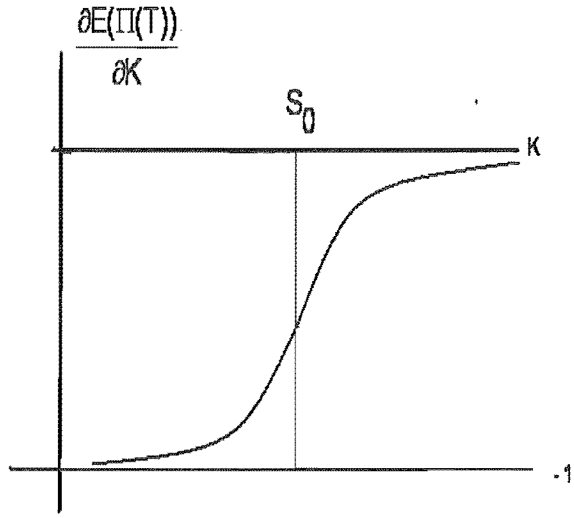


Figure 5.10: Variation of the Expected Payoff with K

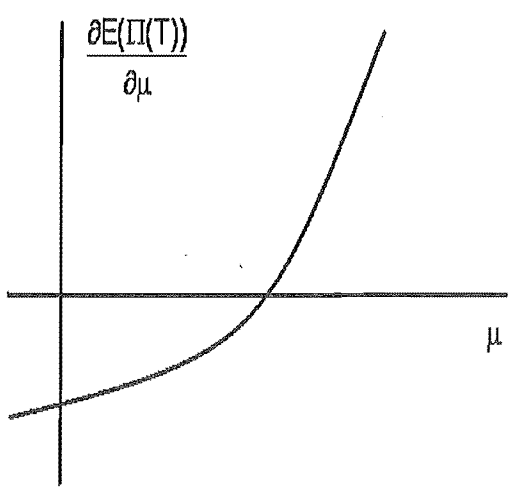


Figure 5.11: Variation of the Expected Payoff with μ

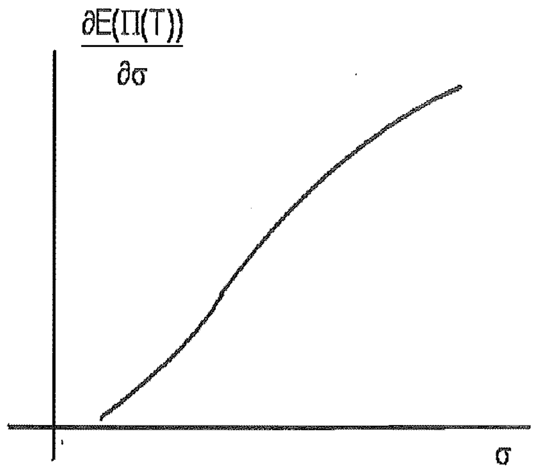


Figure 5.12: Variation of the Expected Payoff with σ

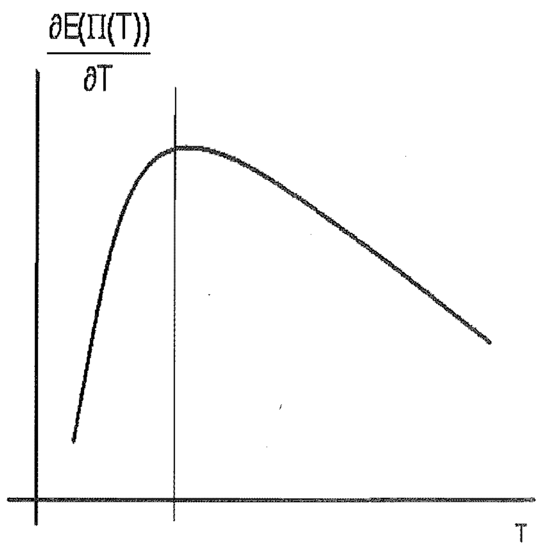


Figure 5.13: Variation of the Expected Payoff with T

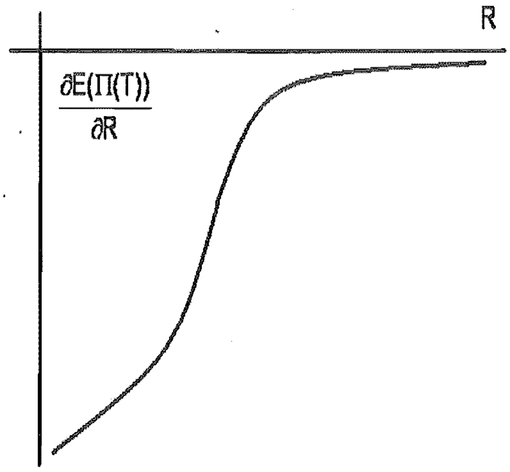


Figure 5.14: Variation of the Expected Payoff with R= $\frac{K}{S_0}$

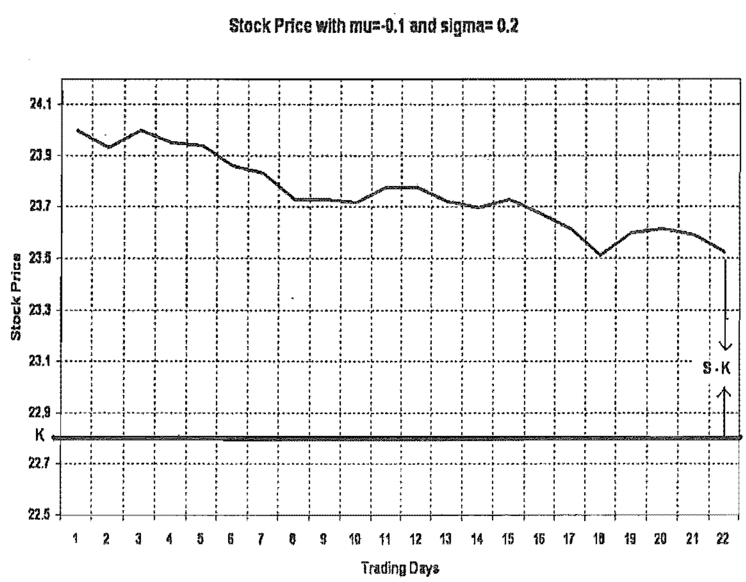


Figure 5.15: Example with negative mu and R less than 1.0

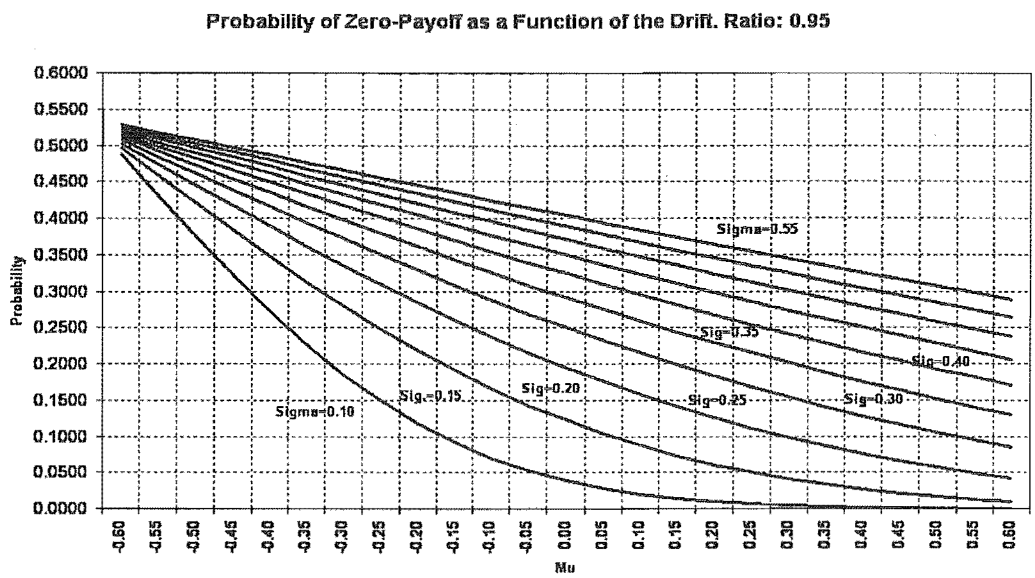


Figure 5.16: Probability of Zero-Payoff as a Function of the Drift, R=0.95

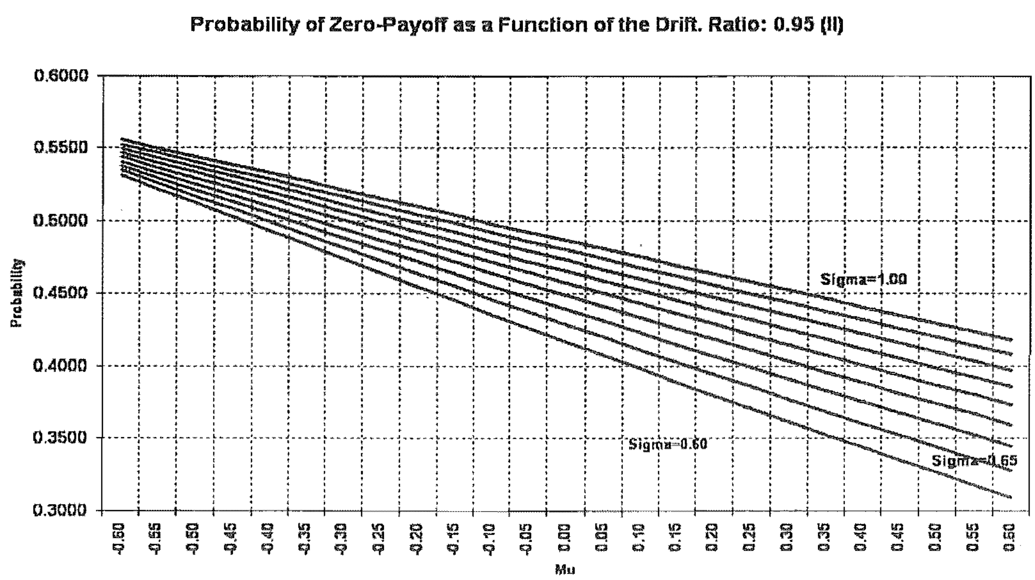


Figure 5.17: Probability of Zero-Payoff as a Function of the Drift (2), R=0.95

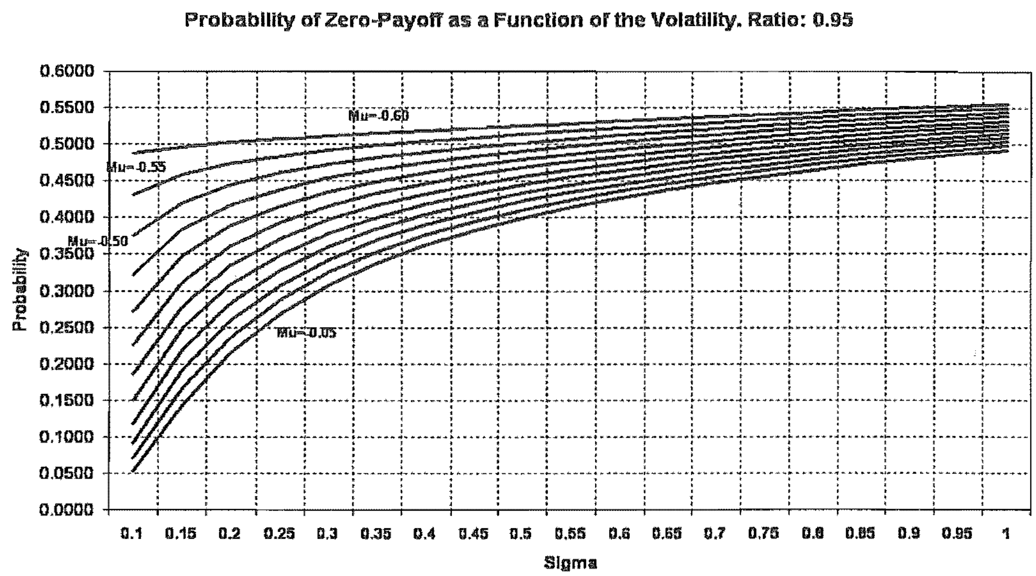


Figure 5.18: Probability of Zero-Payoff as a Function of the Volatility, R=0.95

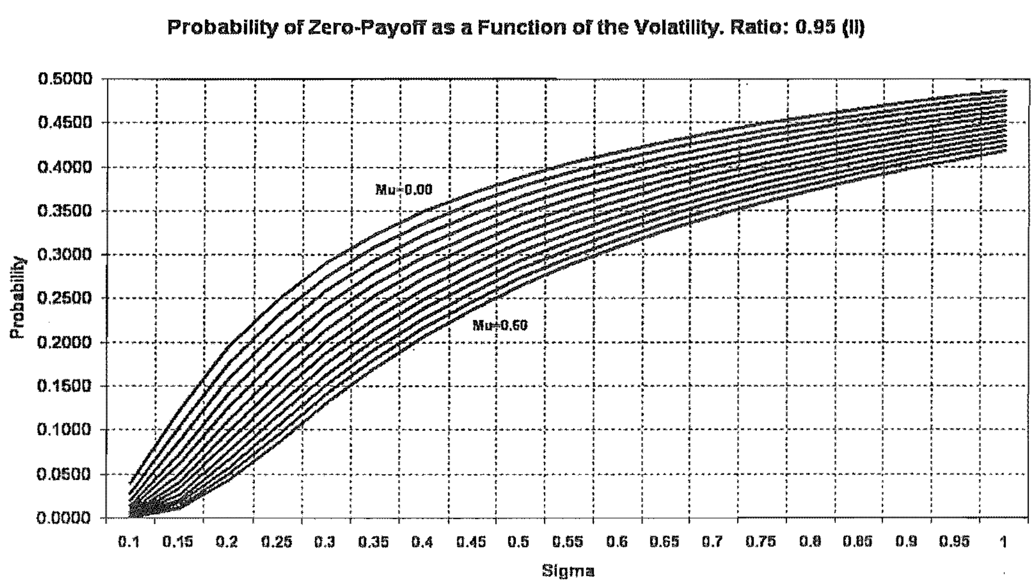


Figure 5.19: Probability of Zero-Payoff as a Function of the Volatility (2), R=0.95

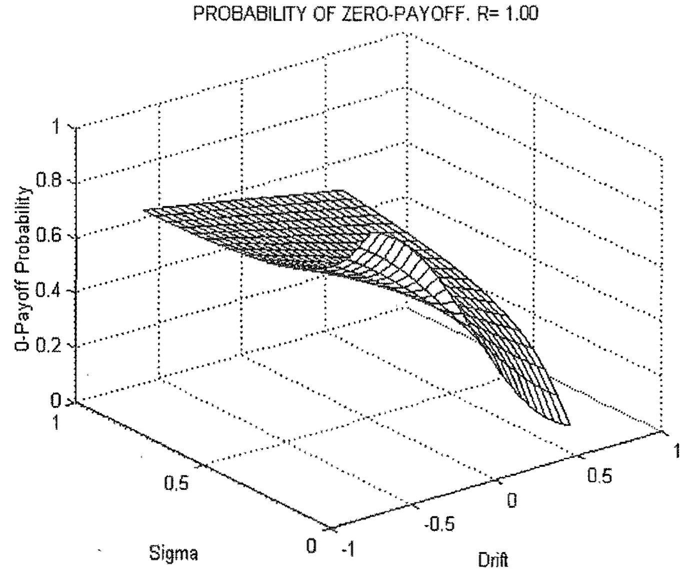


Figure 5.20: Probability of Zero-Payoff, R=1.00

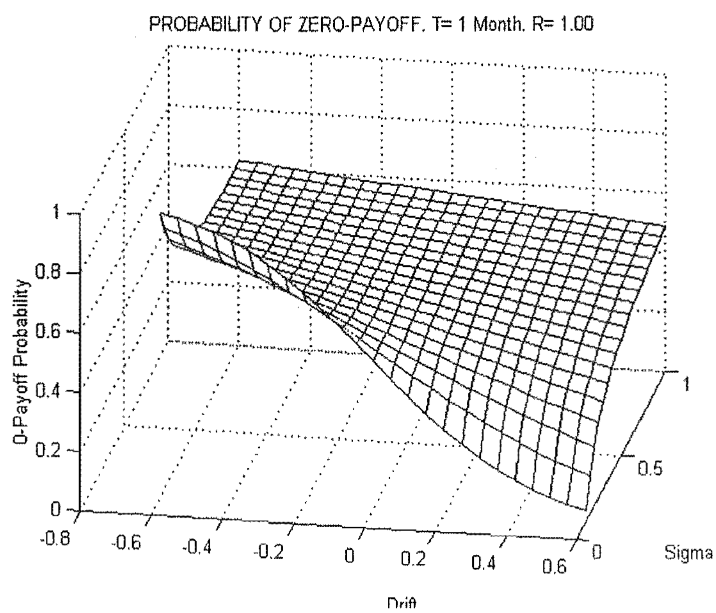


Figure 5.21: Probability of Zero-Payoff, R=1.00

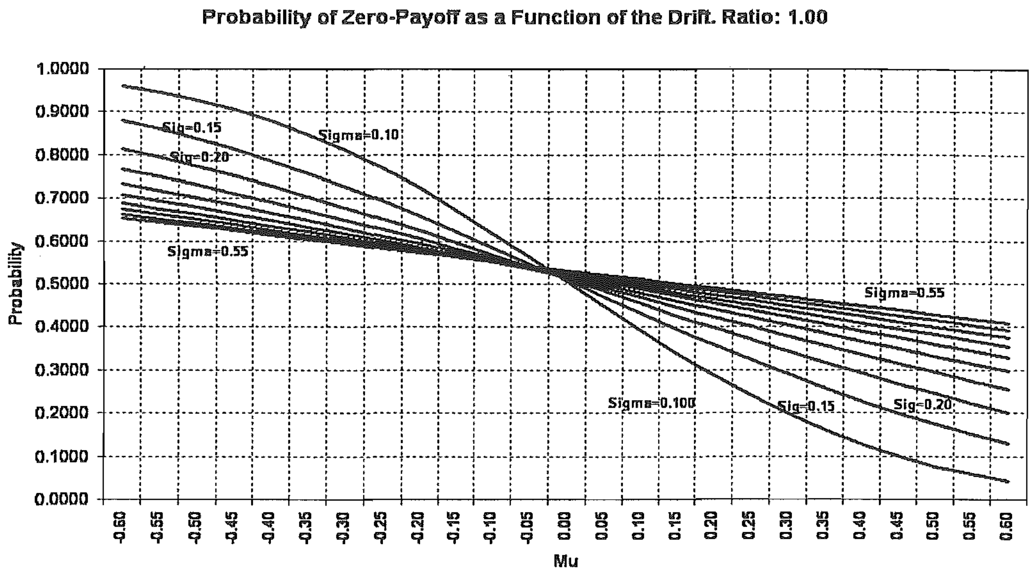


Figure 5.22: Probability of Zero-Payoff as Function of the Drift, R=1.00

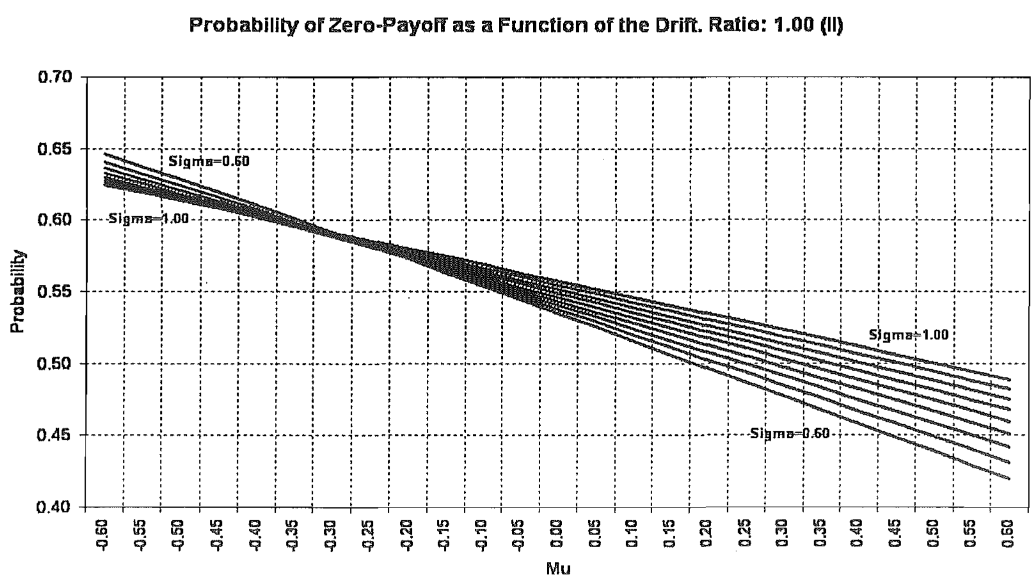


Figure 5.23: Probability of Zero-Payoff as a Function of the Drift (2), R=1.00

Chapter 6

Conclusions

6.1 Seasonality of the VIX

The results of the different test show that the VIX has weekly seasonality expressed as a Friday effect in both differences and returns. This effect seems not be related to causes diffrent from investor's practices and it is stronger during the bull market.
It remains to investigate which types of practice are the most important ones. Other seasonal effects that appear when the OLS method is used are caused more probably by the correlation and the heteroscedasticity of the VIX index.

6.2 The Relationship between the VIX and the OEX indices

There is a clear empirical relation between the level of the VIX and the minimun returns, at three-month, six-month and one-year horizons. High levels of the VIX are related with high minimum returns.

Also, high levels of the VIX are related with high mean returns.

Under the assumption that the market will behave the same way it would be wise to buy OEX options, when the the VIX is high.

Explain the above mentioned phenomena could explain the success of many contrarian investors, since high risk perceived by investors seems to implied lower real risk.

If the observed relationship between VIX and OEX continues consistently it could mean that the Efficient Market Hypothesis does not applies to minimum returns or to times of high fear in the market : It would be possible to take advantage of the times of high fear in the market and get always positive returns for three-month and six-month contrancts on the OEX.

The real causes of this relationship remain to be discovered. Some mathematical tools that could be useful for modelling the observed relations VIX-OEX are fully develeped just for Gaussian processes and are not appliable to the VIX. Although new developments regarding barrier crossing of non-gaussian processes could have been published I do not know them. Future research in this field will benefit greatly of that kind of works.

6.3 The New VIX

By the time in which the second part of this project was done (September 2003), the CBOE changed the methodology of calculating the VIX. Now it is calculated from a different range of strike prices (not only at-the-money) options on the S&P 500. Therefore, these findings do not apply to this new VIX. However, the CBOE will continue calculating and maintain the original VIX. That 'old' index will appear under the ticker symbol "VXO"; this will allow continuing doing research and testing the robustness of the findings here exposed.

6.4 The Option's Payoff Distribution

A speculator or a trader in isolation can use this distribution to value his/her probabilities of zero-payoff and to estimate the expected payoff when he/she has estimates of the future drift and consideres that the B-S model reasonably

approximate the situation he/she experiences.
Closely related to the distribution of option's payoff it is the probability of options expiring in the money. By adequately handlig the information provided in this part of the thesis, this probability can be estimated under the B-S assumptions.

The study of this topics out of the B-S model requires numerical methods and simulation and should be attempted if the assupmtions of the model seem not appropriate.

Bibliography

[1] Amaral, Luis A. N. et al. The distribution of returns of stock prices. *International Journal of Theoretical and Applied Finance.* 3(3):365-369.

[2] Aparicio, Felipe M. and Estrada, Javier. Empirical distributions of stock returns: European securities markets, 1990-95. *European Journal of Finance.* 7:1-21.

[3] Ariel, Robert A.(1990).High Stock Returns Before Holidays: Existence and evidence on possible causes. *The Journal of Finance.* 49(5):1611-1626.

[4] rsad, Zainudin and Coutts (1997), J. Andrew. Security price anomailies in the London international stock exchange: A 60 Year Perspective. *Applied Financial Economics.* 7:455-464.

[5] Bell, William R. and Hillmer, Steven C. (1984) Issues iInvolved with the seasonal adjustment of economic time series. *Journal of Business & Economic Statistics.* 2(4) :98-127.

[6] Bailey, Warren and Stulz, Rene (1989). The pricing of stock index options in a general equilibrium model. *Journal of Financial and quantitative Anlaysis.* 24(1):1-12.

[7] Banz, R. (1981) The Relationship Between Returns and Market value of Common Stocks. *Journal of Financial Economics* 9: 3-18 .

[8] Barndorff-Nielsen, Ole E. and Shephard, Neil. (2002). Econometric analyisis of realized volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society.* 64(2):253-280.

[9] Bell, William R. and Hillmer, Steven C. (1984) Issues iInvolved with the seasonal adjustment of economic time series. *Journal of Business & Economic Statistics*. 2(4) :98-127.

[10] Bidarkota, Prasad V. and McCullogh J. Huston. (1996). Real stock returns: non-normality, seasonality, and volatility persistence, but not predictability. Working paper. Department of Economics, The Ohio State University.

[11] Björk, Thomas. (2004) *Arbitrage Theory in Continuous Time*. Second edition. Oxford University Press. Ney York.

[12] Bollerslev, Tim and Mikkelsen, Hans Ole (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*. 73:151-184.

[13] Boness, A, J. (1964). Elements of a theory of stock option value. *Journal of Political Economy*. 72(2):163-75.

[14] Brockman, Paul and Chowdhury, Mustafa(1997) Deterministic versus stochastic volatility: Implications for Option Pricing Models. *Applied Financial Economics*. 7: 499-505.

[15] Cox, John C., et Al. (1979). Option pricing: a simplified approach. *Journal of Financial Economics* 7:229-263.

[16] Christensen, B.J. and Prabhala, N.R.(1998). The Relation between implied and realized volatility. *Journal of Financial Economics* 50:125-150.

[17] Cubadda, Gianluca. (2001) Common features In time series with both deterministic and stochastic seasonality. *Econometric Reviews*. 20(2) :201-216.

[18] Daigler, Robert T.(1994). *Financial Futures and Options*.New York. Harper College Publishers.

[19] Dimson, E. editor. (1988) *Stock market anomalies*. Cambridge University Press. Cambridge.

[20] Dimson, E. and Marsh, P. (1999) Murphy's Law and market anomalies. *The Journal of Portfolio Management* winter: 53-69.

[21] Durbin, J. and Kenny, P.B. (1979). *Seasonal analysis of economic time series* A. Zellner, ed. U.S. GPO, Washington DC, pp. 173-187.

[22] Enders, Walter (1995). *Applied Econometric Time Series.* John Wiley &Sons. New York.

[23] Feisntein, S. (1989).The Black-Scholes formula is nearly linear in s for at-the- money options: therefore implied volatilities from at-the-money options are virtually unbiased. Working Paper, Jones Graduate School of Administration, Rice University.

[24] Fleming, Jeff and Whaley Robert.(1994). The value of wildcard options *The Journal of Finance.* 49(1):215-236.

[25] Fleming, Jeff et Al.(1995) Predicting stock market volatility: a new measure. *The Journal of Future Markets.* 15(3):265-302.

[26] Fleming, Jeff .(1997) The quality of market volatility forecasts implied by S&P100 index option prices. Working paper. Rice University.

[27] Fleming, Jeff .(1998) The economic significance of the forecast bias of S&P100 index option implied volatility. Working paper. Rice University.

[28] Frances, Philip Hans, and Paap Richard (2000). Modelling day-of- the-week seasonality in the S&P500. *Applied Financial Economics.* 10:483-488

[29] French, Kenneth. (1980) .Stock returns and the weekend effect. *Journal of Financial Economics.* 8: 55-69.

[30] French, Kenneth and Roll, Richard (1986). Stock return variances. The arrival of iInformation and the reaction of the traders.*Journal of Financial Economics.* 17:5-26.

[31] Fountas, Stilianos and Segredakis, Konstantinos N. Emerging stock markets return seasonalities: the January effect and the tax-loss selling hypothesis. Working paper No. 37. Department of Economics. National University of Ireland, Galway.

[32] Galai, Dan. On the Boness and Black-Scholes models for valuation of call options. *Journal of Financial and Quantitalite Analysis.*March-1978: 15-27.

[33] Geman, Hélyette et al. (1995).Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability.* 32:443-458.

[34] Gultekin, Mustafa N. and Gultekin N. Bulent (1983). Stock market seasonality: International evidence. *Journal of Financial Economics.* 12:469-81.

[35] Haefke, Christian (2002). Index forecasting and model selection. *International Journal of Intelligent Systems in Accounting, Finance & Management.* 11 :110-135.

[36] Harasty, Hèléne and Roulet, Jacques(2000). Modeling stock market returns. An error correction model. *The Journal of Portfolio Analysis.* Winter-2000:33- 46.

[37] Howison et al. editors (1995). *Mathematical models in finance.* Chapman & Hall. London.

[38] Harris, Richard D.F. and Kücúkózmen, Coskun.(2001). The empirical distribution of stock returns: Evidence from an Emerging European Market. *Applied Economics Letters* 8:367-371.

[39] Hylleberg, Svend, editor. (1992).*Modelling seasonality.* Oxford University Press.

[40] Jarrow, robert and Turnbull, Stuart (1994). *Derivative Securities.* Cincinnati, Ohio. South-Western College Publishing.

[41] Jokivoulle, Esa(1995.). Pricing european options on autocorrelated indexes. *The Journal of Derivatives.* Winter 1998:39-52.

[42] Klemosky, Robert C. The impact of options expirations on stock prices. *Journal of Financial and Quantitalite Analysis.*September-1978: 507-519.

[43] Kolb, Robert W. (1996). *Futures, Options and Swaps.* Cambridge, Massachusetts. Kolb Publishing Company Inc.

[44] Lakonishok, J. and Maberly E. (1986). The Weekend Effect: Trading Patterns of Individual and Institutional Investors. *The Journal of Finance* 45(1):231-244.

[45] Levy, Haim and Yoder, James A.(1996). A sotchastic approach to evaluating alternative estimators of the variance for use in the Black-Scholes option pricing model. *Applied Financial Economics.*6:377-382.

[46] Nowman, K. Ben and Wang Helen (2001) . Modelling commodity prices using continuous time models. *Applied Economic Letters.*8:341-345.

[47] Officer, R. (1975) .Seasonality in the australian capital market: Market efficiency and empirical iIssues. *Journal of Financial Economics.* March: 29-52.

[48] Poon, Ser-Huang and Granger, Clive (2001). Forecasting Financial Market Volatility. A Review. Working Paper. University of California, san Diego.

[49] Reinganum, M. (1981) Misspecification of capital asset pricing: Empirical anomalies based on earning yields and market values. *Journal of Financial Economics.*March: 19-46.

[50] Ritchken, Peter (1996).*Derivative Markets, Theory, Strategy and Applications.* New York. Harper Collins College Publishers.

[51] Rozeff, M.S.and Kinney, W.R. Jnr (1976). Capital market seasonality: The case of stock returns.*Journal of Financial Economics.*3: 379-402.

[52] Saez Marc (1997). Option pricing under stochastic volatility and sochastic iInterest rate in the Spanish case. *Applied Financial Economics* 7:379-394.

[53] Sias, Richard W. and Starks, Laura T (1995). The day-of-the week anomaly: The role of institutional investors. *Financial Analysts Journal.* May-June:58-67.

[54] Smith, Clifford W. Jr.(1976). Option pricing. A review. *Journal of Financial Economics* . 3:3-51.

[55] Solnik, Bruno. (1990). The distribution of daily stock returs and settlement procedures: The paris bourse. *The Journal of Finance* 45(5): 1601-1609.

[56] Stoll, hans R. and Whaley, Robert E.(1990) The dynamics of stock index and stock index futures returns. *Journal of Financial and Quantitative Analysis.* 25(4):441-468.

[57] Thaler, Richard H (1987). Anomalies. The January effect. *Economic Perspectives.* 1(1):197-201.

[58] Thaler, Richard H (1987). Anomalies. Seasonal movements in security prices II: Weekend, Holiday, Turn of the Month, and Intraday Effects *Economic Perspectives.* 1(1):169-177.

[59] Watanabe, Toshiaki.(2000) Excess kurtosis of conditional distribution for daily stock returns: The Case of Japan. *Applied Economic Letters* 7:353-355.

[60] Wei, Jason Z, (1996) A nonparametric test for monthly seasonality for international stock markets. Working paper, University of Toronto.

[61] Whaley, Robert E. (1993). Derivatives on market volatility: Hedging tools, long overdue. *The Journal of Derivatives.* Fall: 71-84.

[62] Whaley, Robert E. (2000) The investor fear gauge. explication of the CBOE VIX. *The Journal of Portfolio Management.* Spring:13-17.

[63] Wilmott, Paul (1998). *Derivatives. The Theory and practice of financial engineering.* John Wiley & Sons. New York.

ADDITIONAL FIGURES

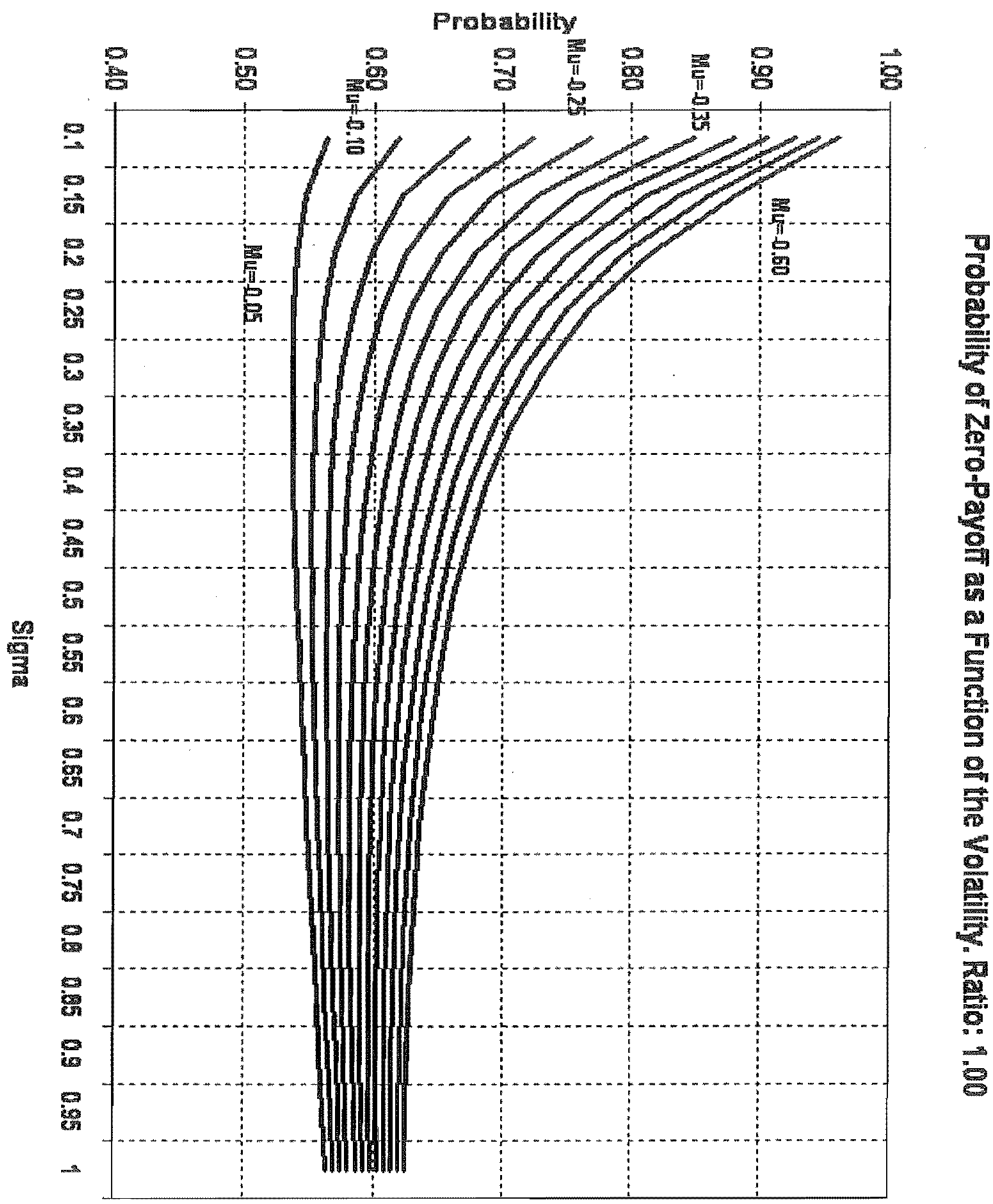


Figure 5.24: Probability of Zero-Payoff, Function of the Volatility , R=1.00

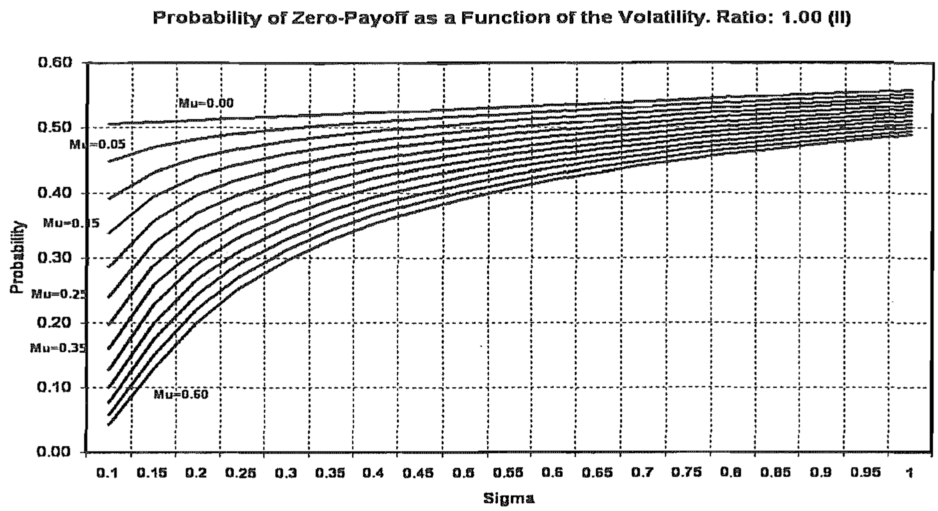


Figure 5.25: Probability of Zero-Payoff, Function of the Volatility(2), R=1.00

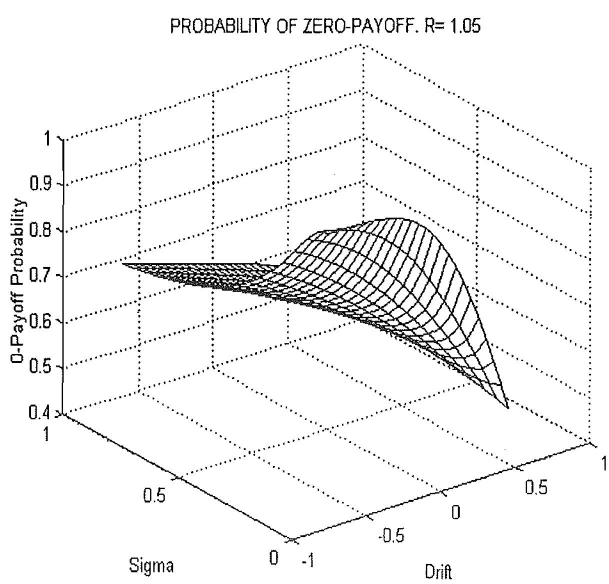


Figure 5.26: Probability of Zero-Payoff, R=1.05

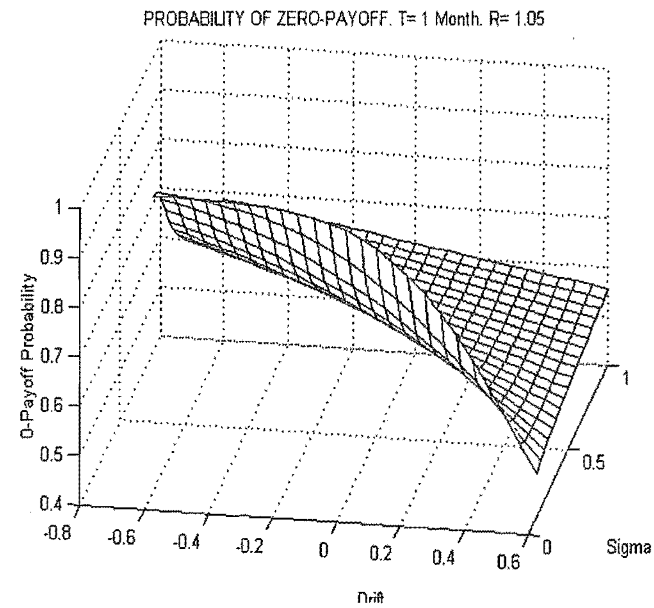


Figure 5.27: Probability of Zero-Payoff, R=1.05

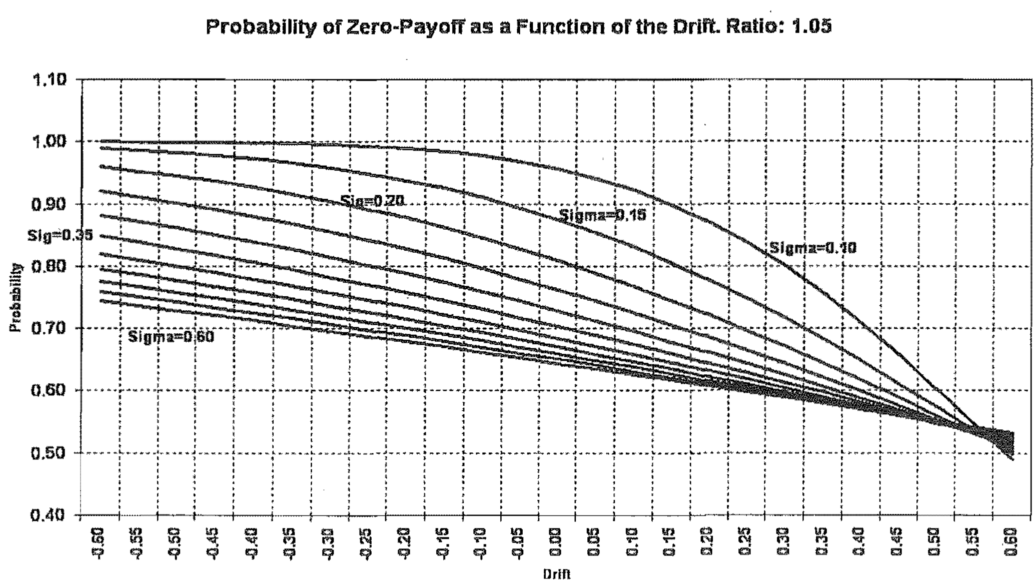


Figure 5.28: Probability of Zero-Payoff as a Function of the Drift (I) R=1.05

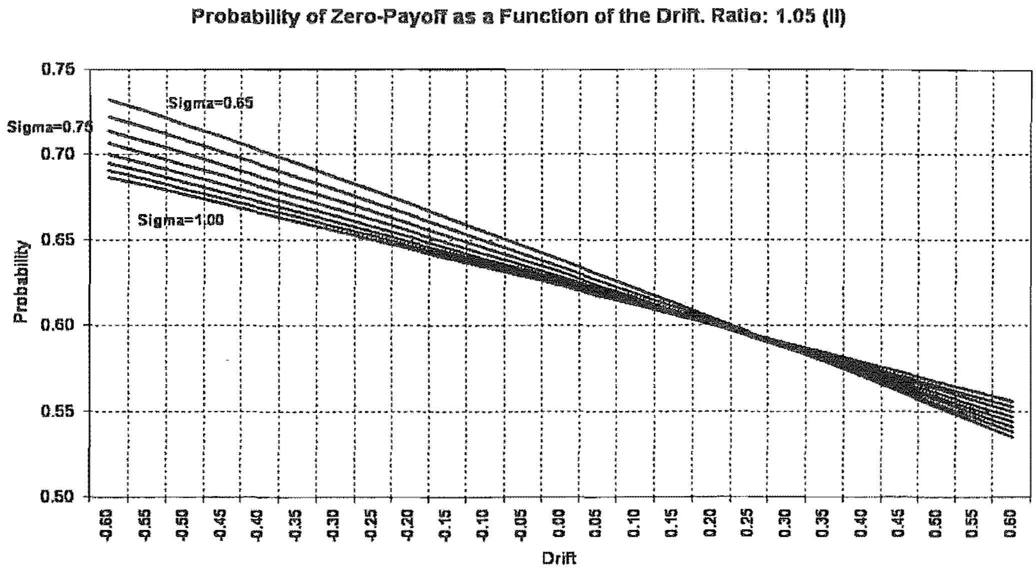


Figure 5.29: Probability of Zero-Payoff as a Function of the Drift (II), R=1.05

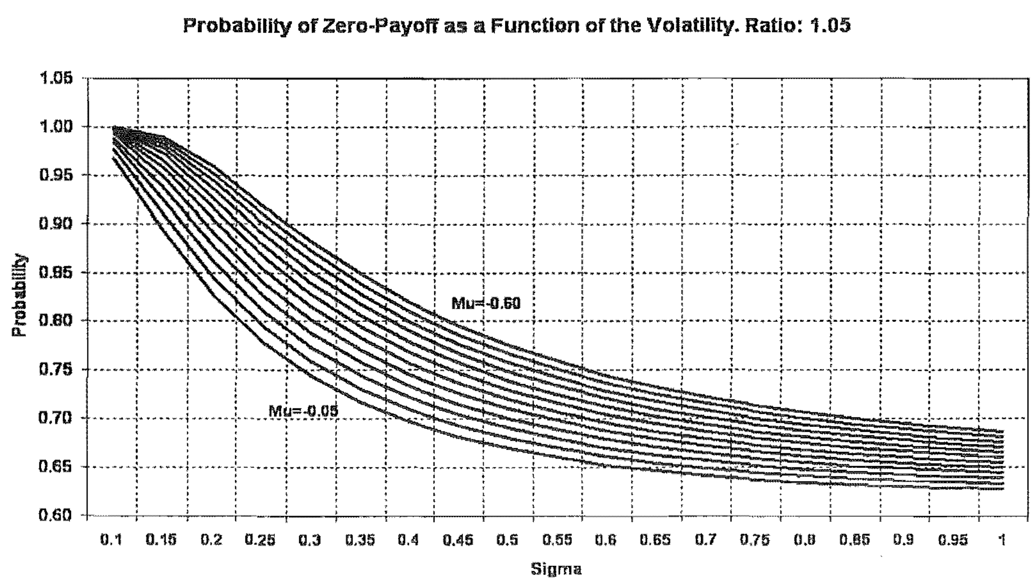


Figure 5.30: Probability of Zero-Payoff as a Function of the Volatility(I) , R=1.05

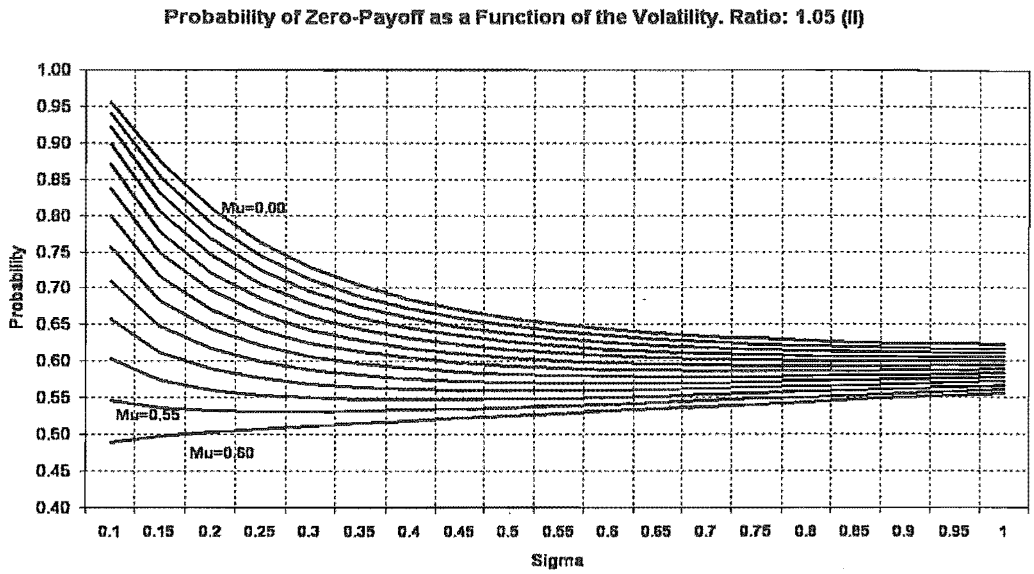


Figure 5.31: Probability of Zero-Payoff as a Function of the Volatility(II), R=1.05

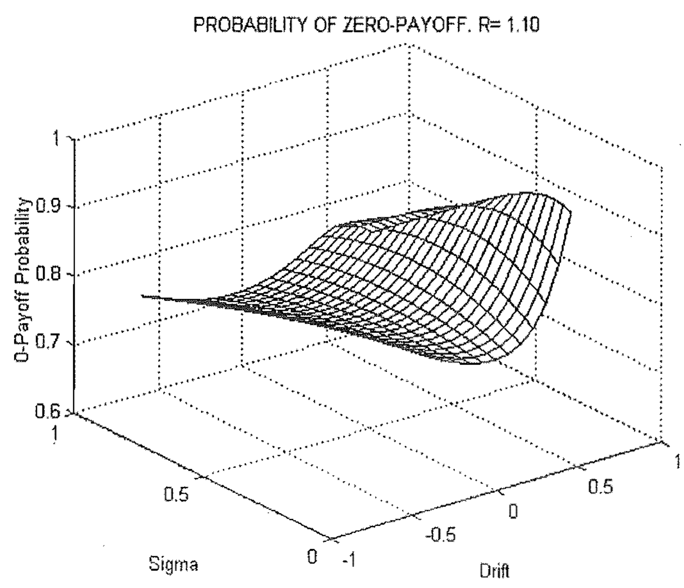


Figure 5.32: Probability of Zero Payoff, R=1.10

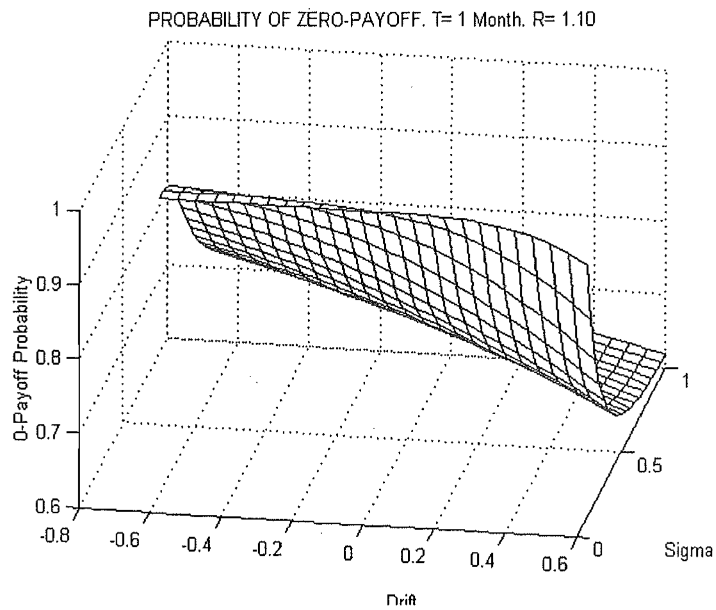


Figure 5.33: Probability of Zero Payoff, R=1.10

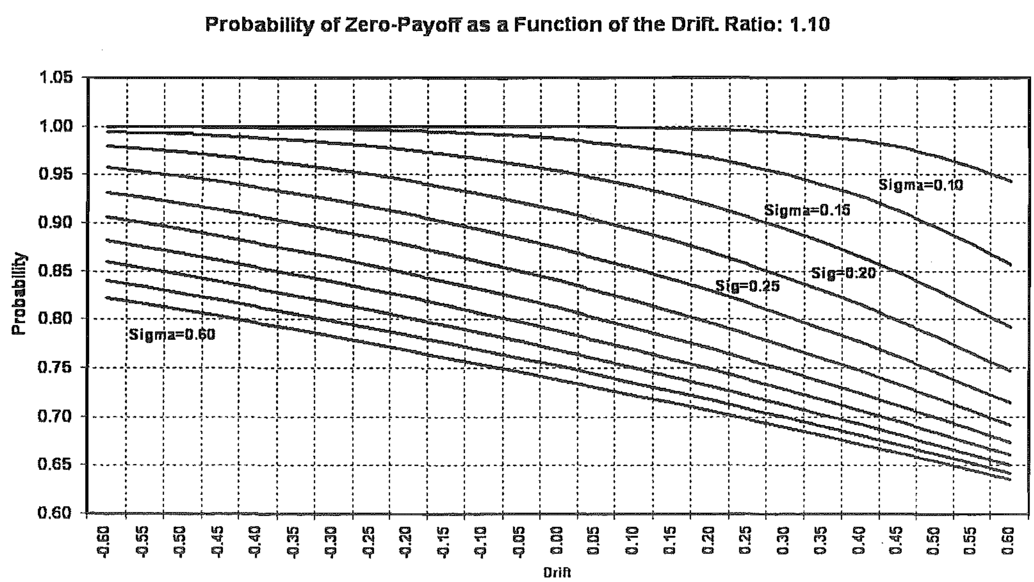


Figure 5.34: Probability of Zero-Payoff as a Function of the Drift (I) R=1.10

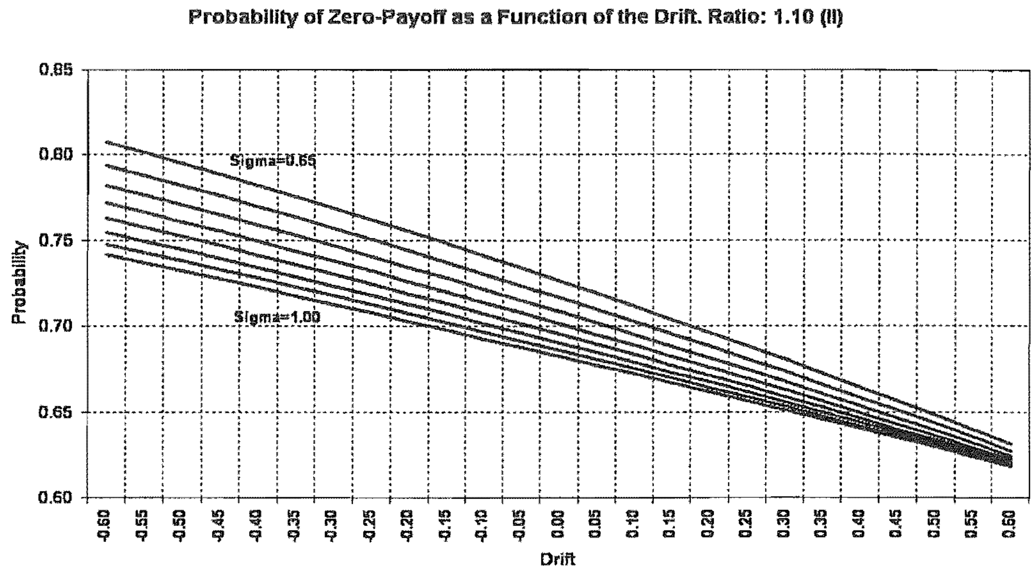


Figure 5.35: Probability of Zero-Payoff as a Function of the Drift (II), R=1.10

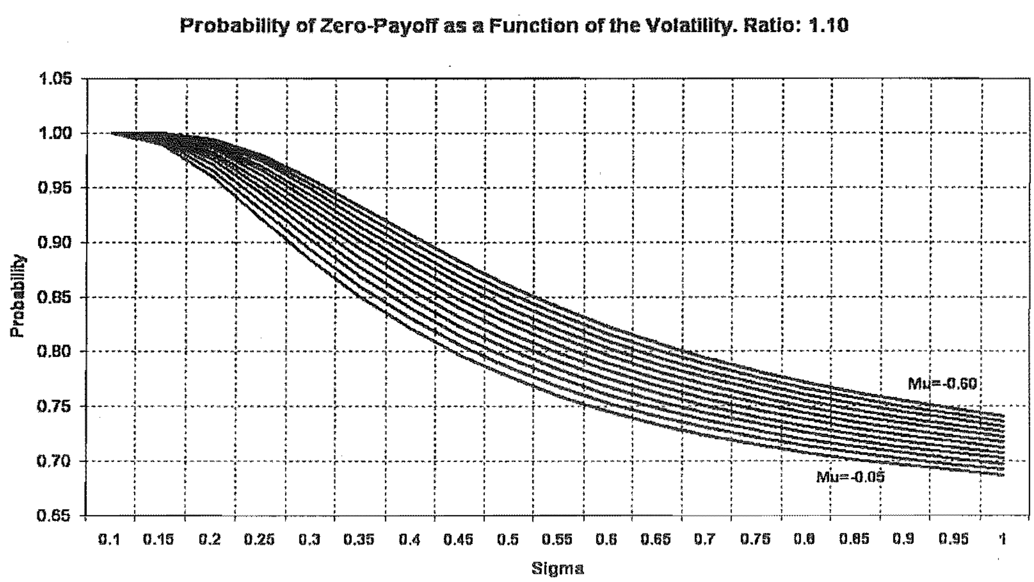


Figure 5.36: Probability of Zero-Payoff as a Function of the Volatility(I) , R=1.10

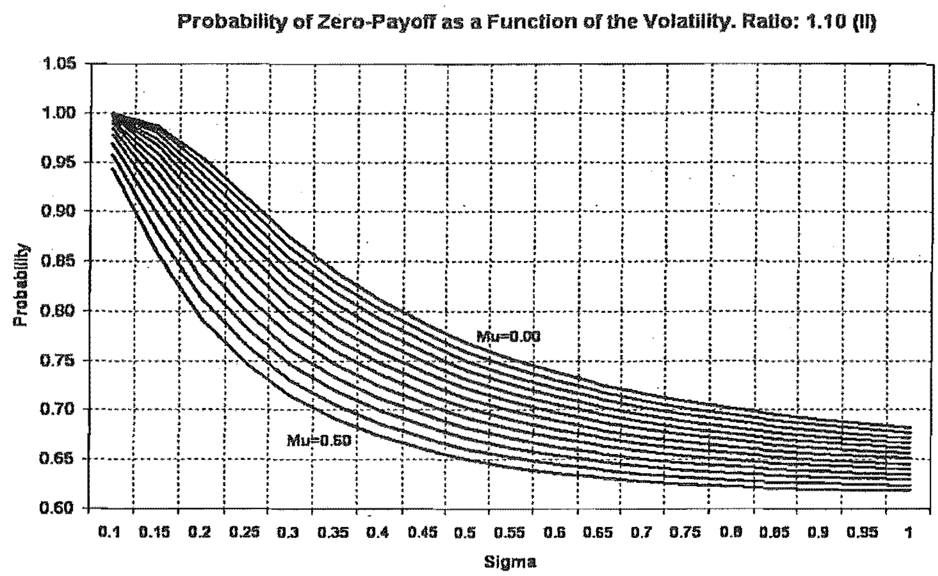


Figure 5.37: Probability of Zero-Payoff as a Function of the Volatility(II), R=1.10

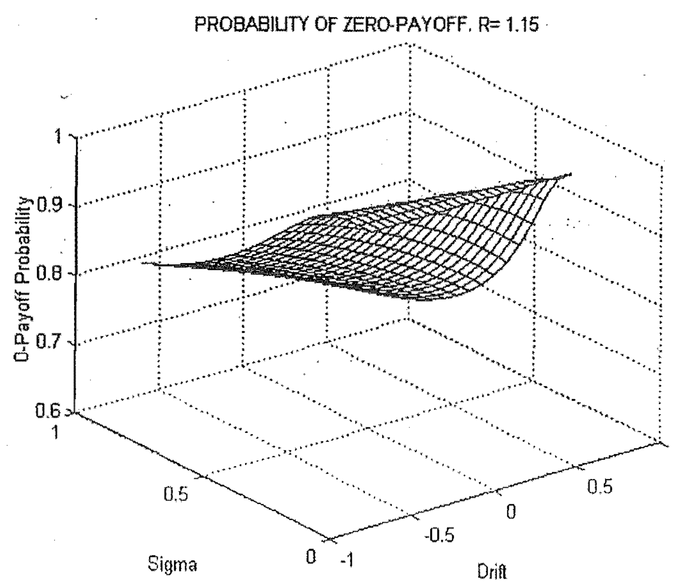


Figure 5.38: Probability of Zero Payoff, R=1.15

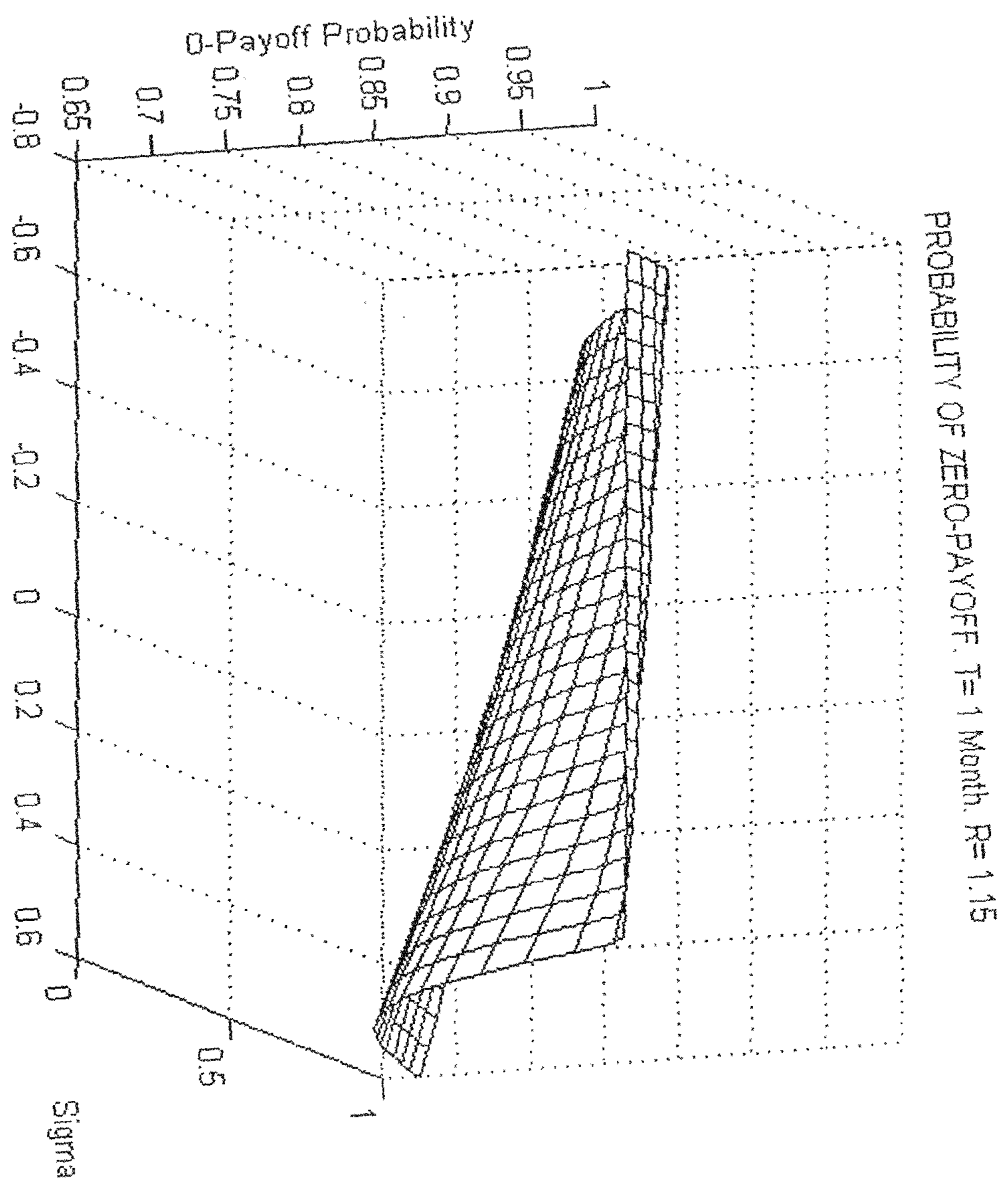


Figure 5.39: Probability of Zero Payoff, R=1.15

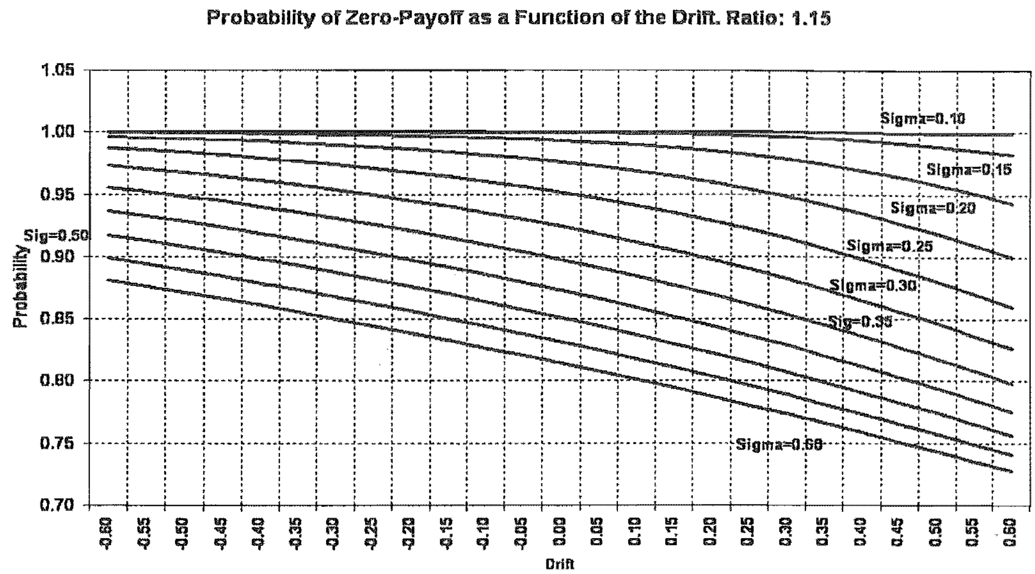


Figure 5.40: Probability of Zero-Payoff as a Function of the Drift. (I) R=1.15

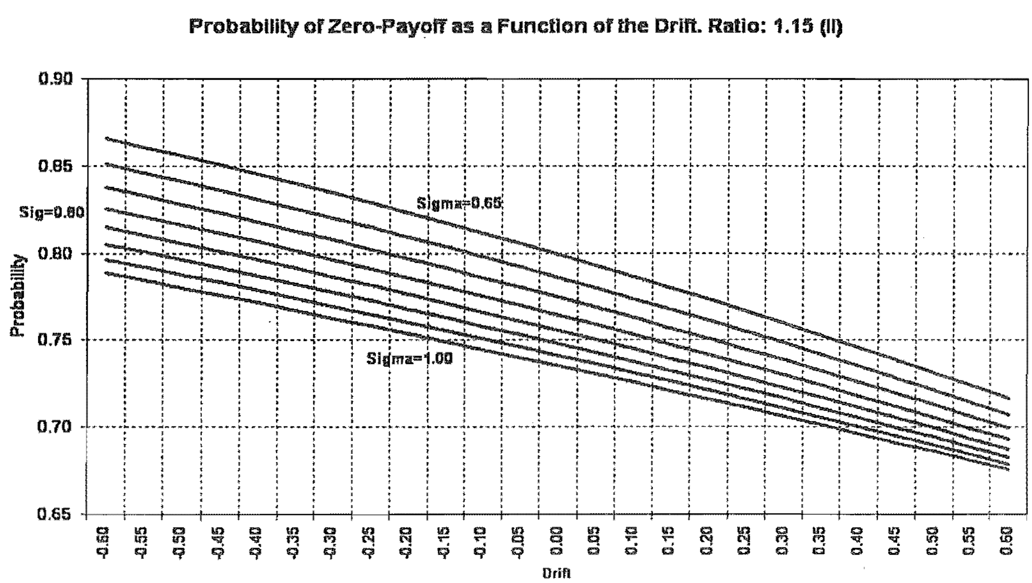


Figure 5.41: Probability of Zero-Payoff as a Function of the Drift (II), R=1.15

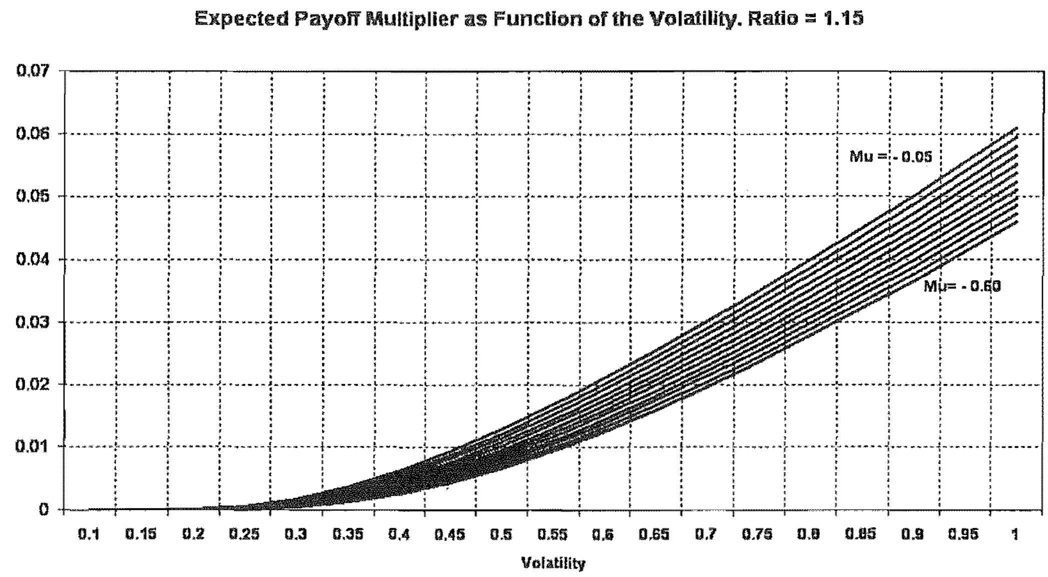


Figure 5.42: Variation of the Expected Payoff with σ (I). R= 1.15

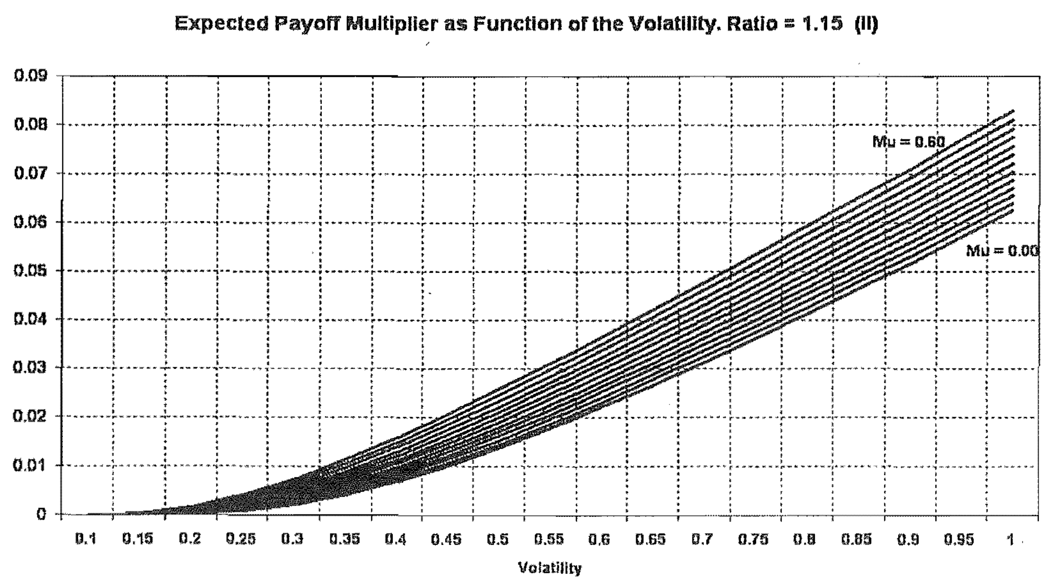


Figure 5.43: Variation of the Expected Payoff with σ (II). R= 1.15

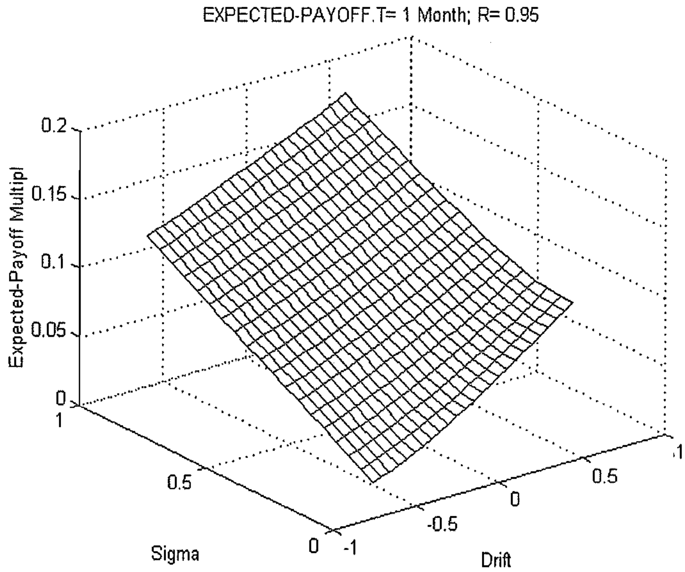


Figure 5.44: Expected Payoff Multipliers for R=0.95

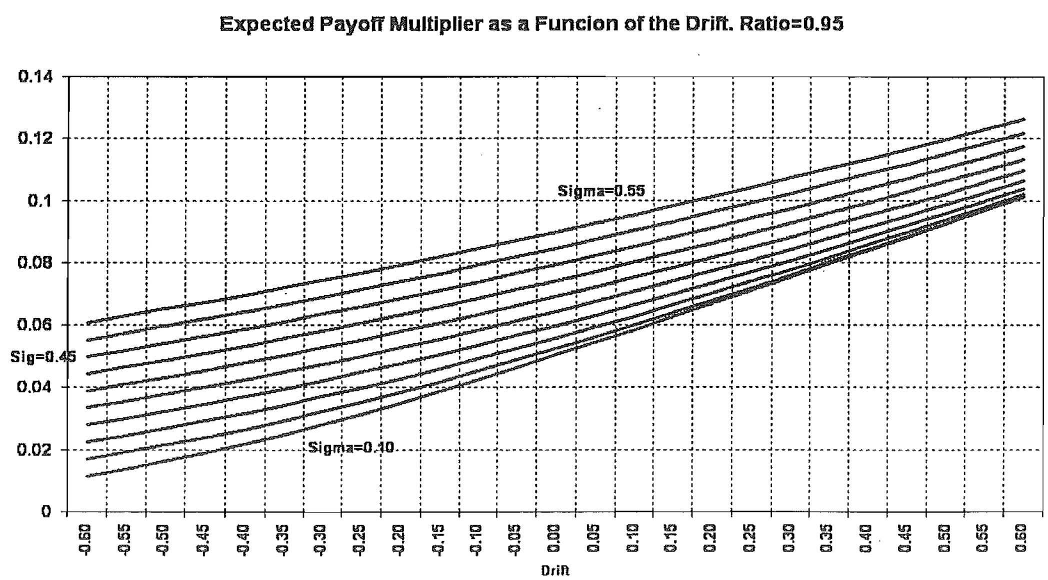


Figure 5.45: Variation of the Expected Payoff with μ(I). R=0.95

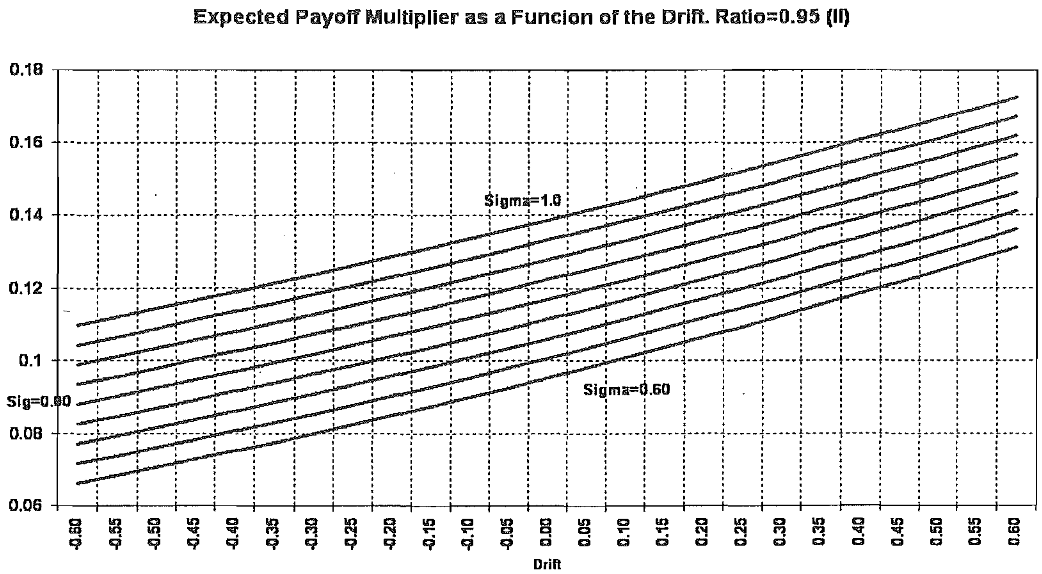


Figure 5.46: Variation of the Expected Payoff with μ. (II) R=0.95

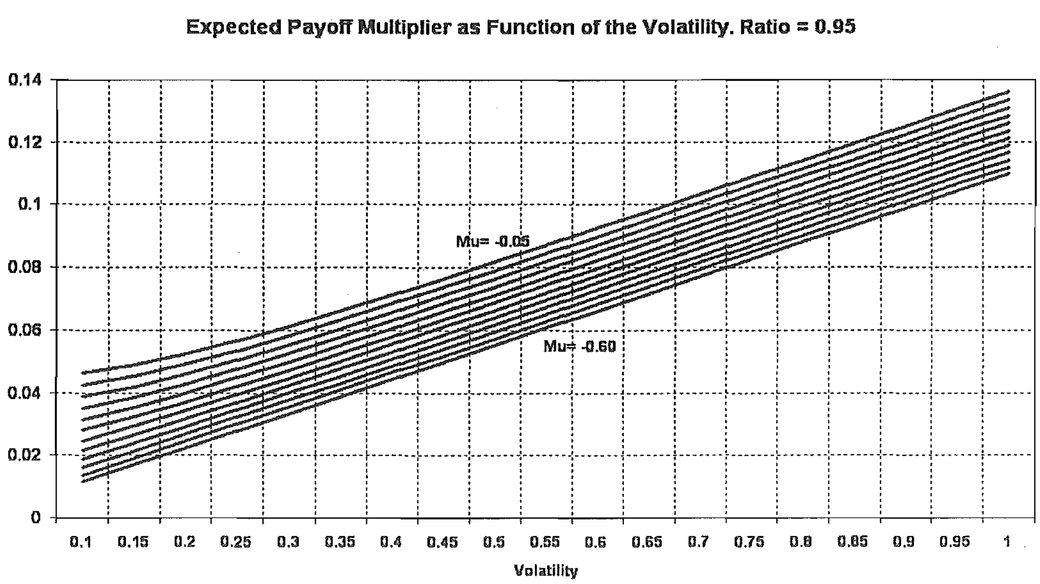


Figure 5.47: Variation of the Expected Payoff with σ. (I) R=0.95

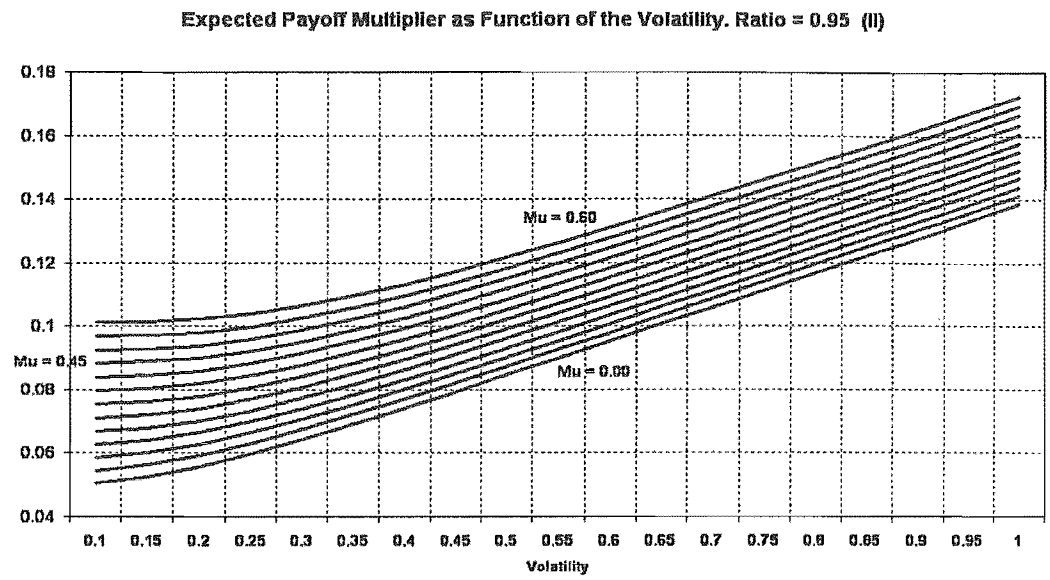


Figure 5.48: Variation of the Expected Payoff with σ. (II) R=0.95

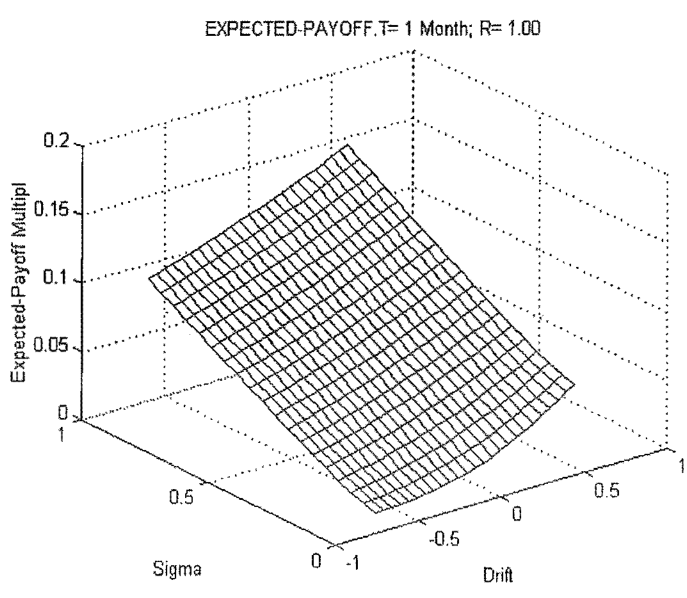


Figure 5.49: Expected Payoff Multipliers, R= 1.00

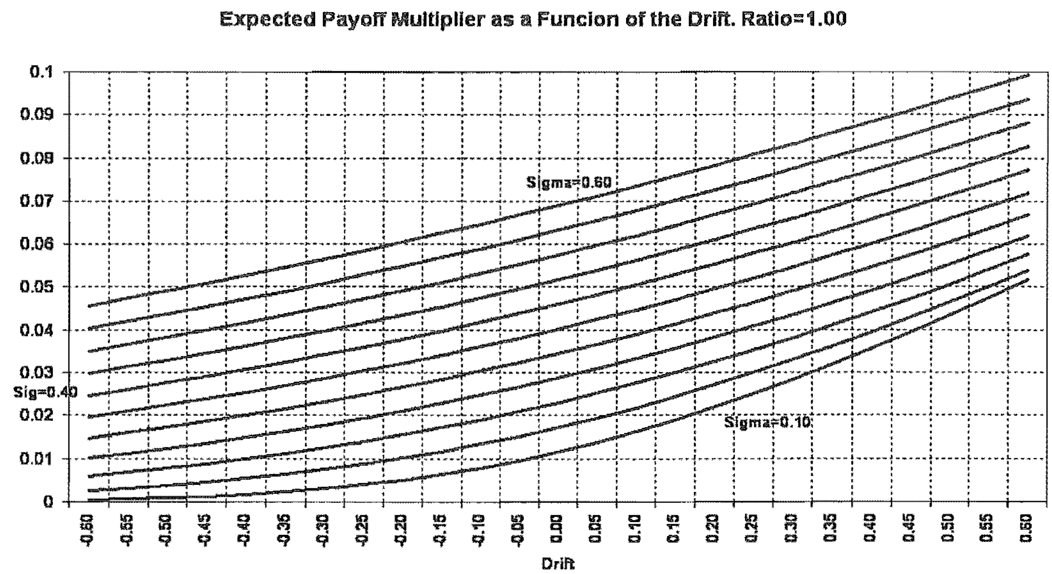


Figure 5.50: Variation of the Expected Payoff with μ (I). R=1.00

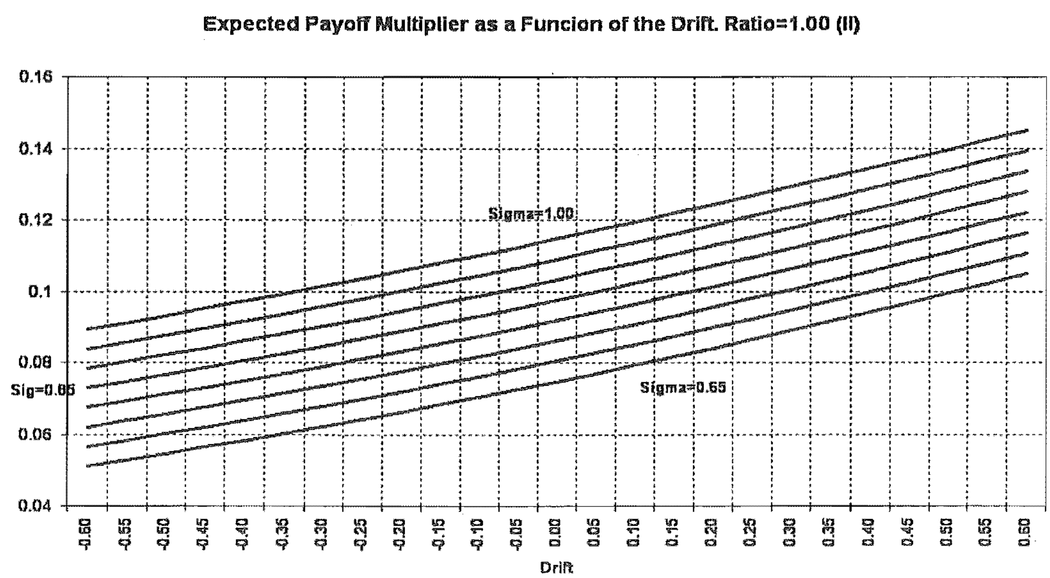


Figure 5.51: Variation of the Expected Payoff with μ (II). R=1.00

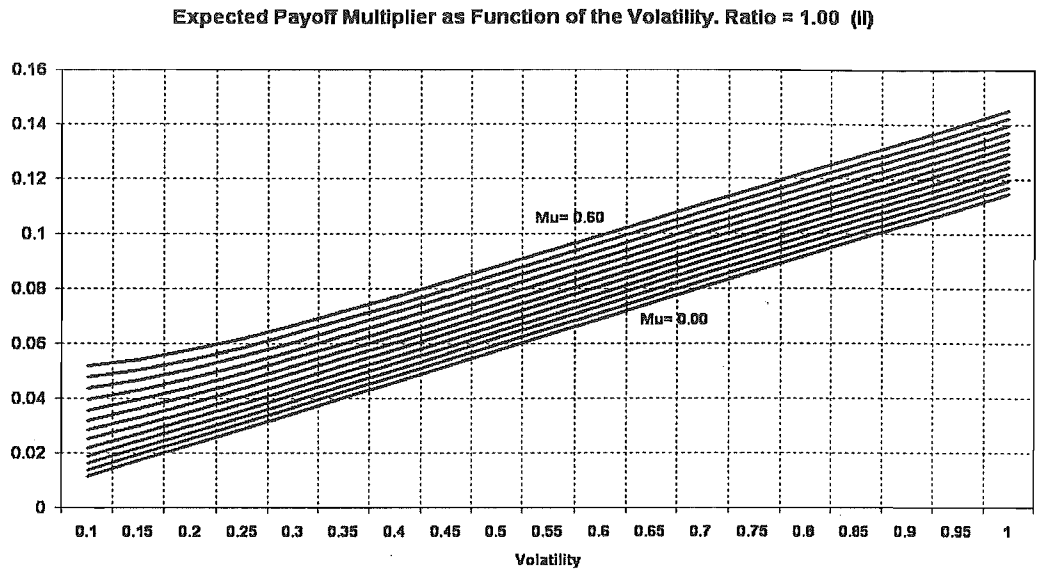


Figure 5.52: Variation of the Expected Payoff with σ(I) R= 1.00

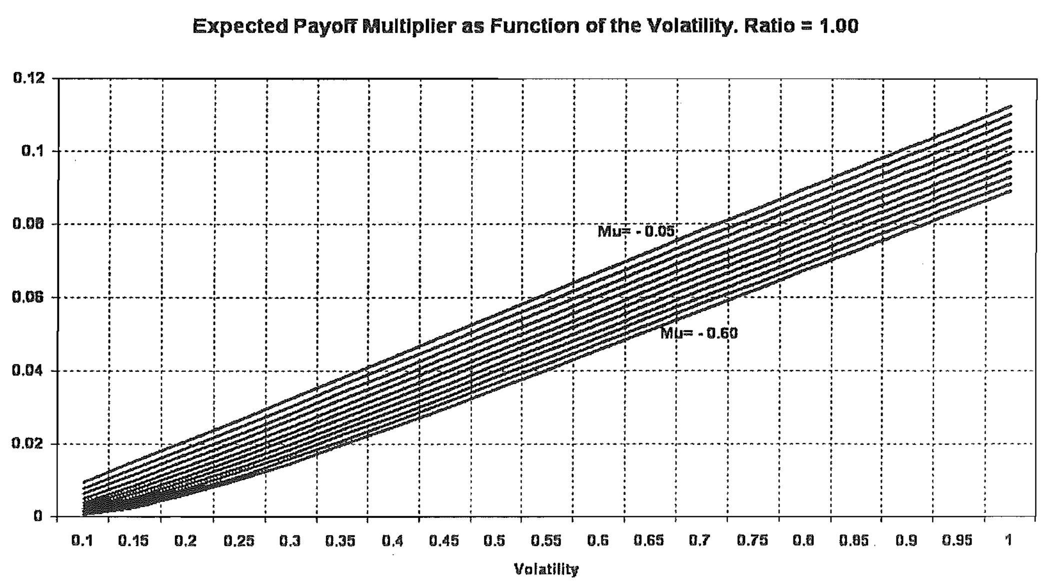


Figure 5.53: Variation of the Expected Payoff with σ (II) R=1.00

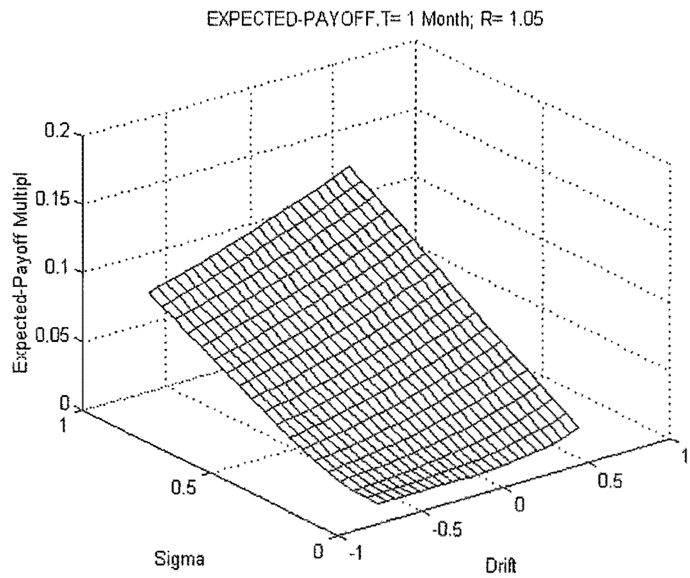


Figure 5.54: Expected Payoff Multipliers, R= 1.05

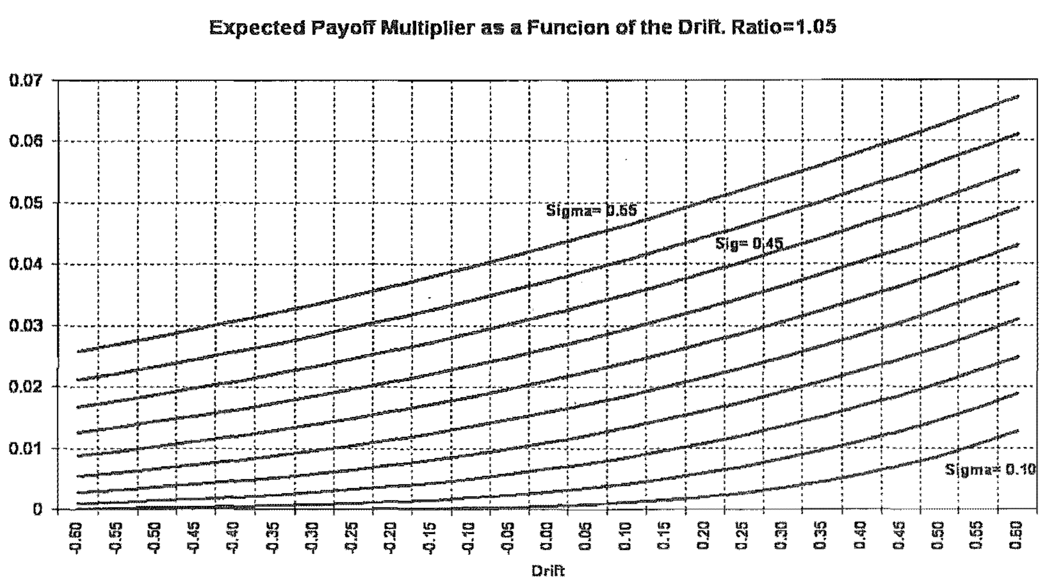


Figure 5.55: Variation of the Expected Payoff with μ (I). R= 1.05

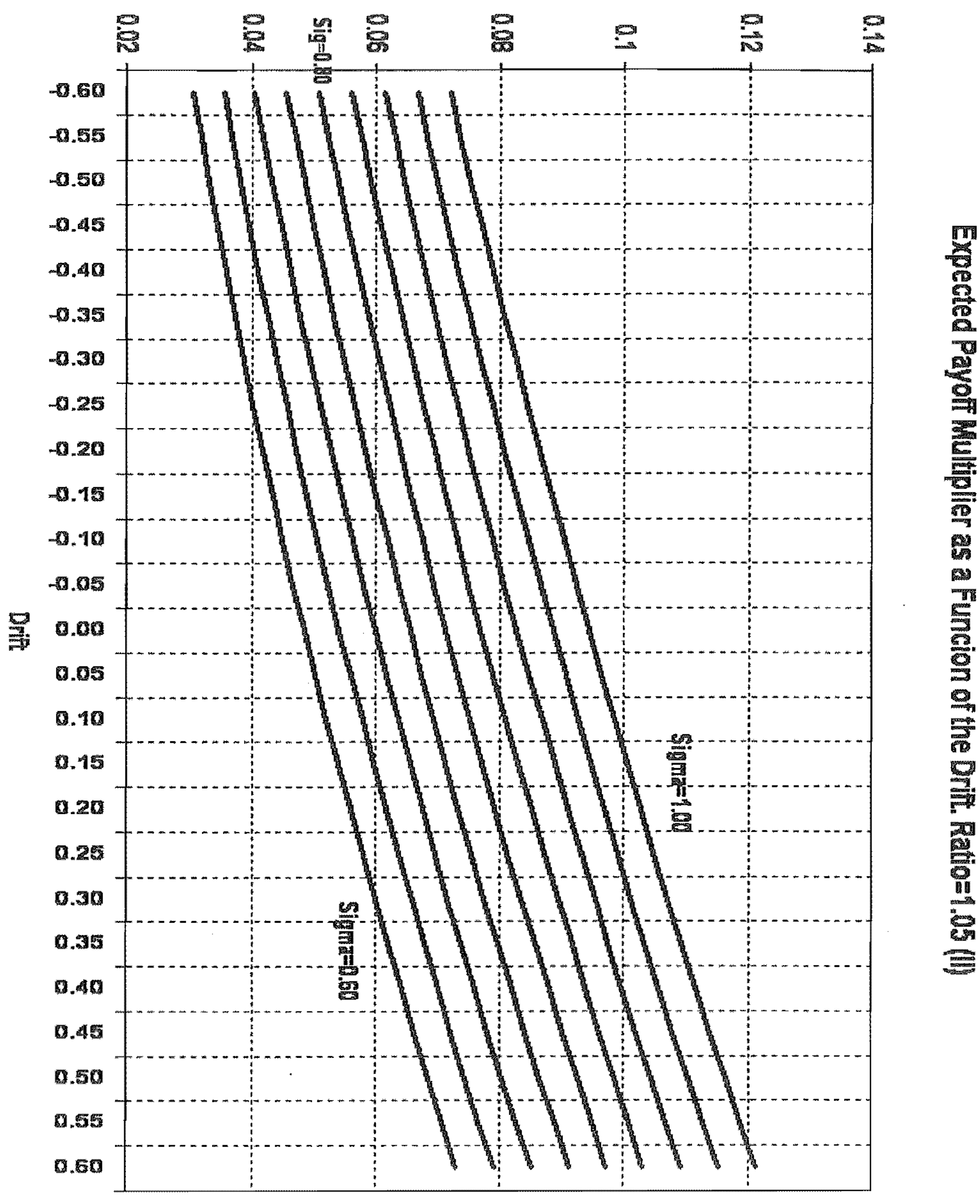


Figure 5.56: Variation of the Expected Payoff with μ (II). R= 1.05

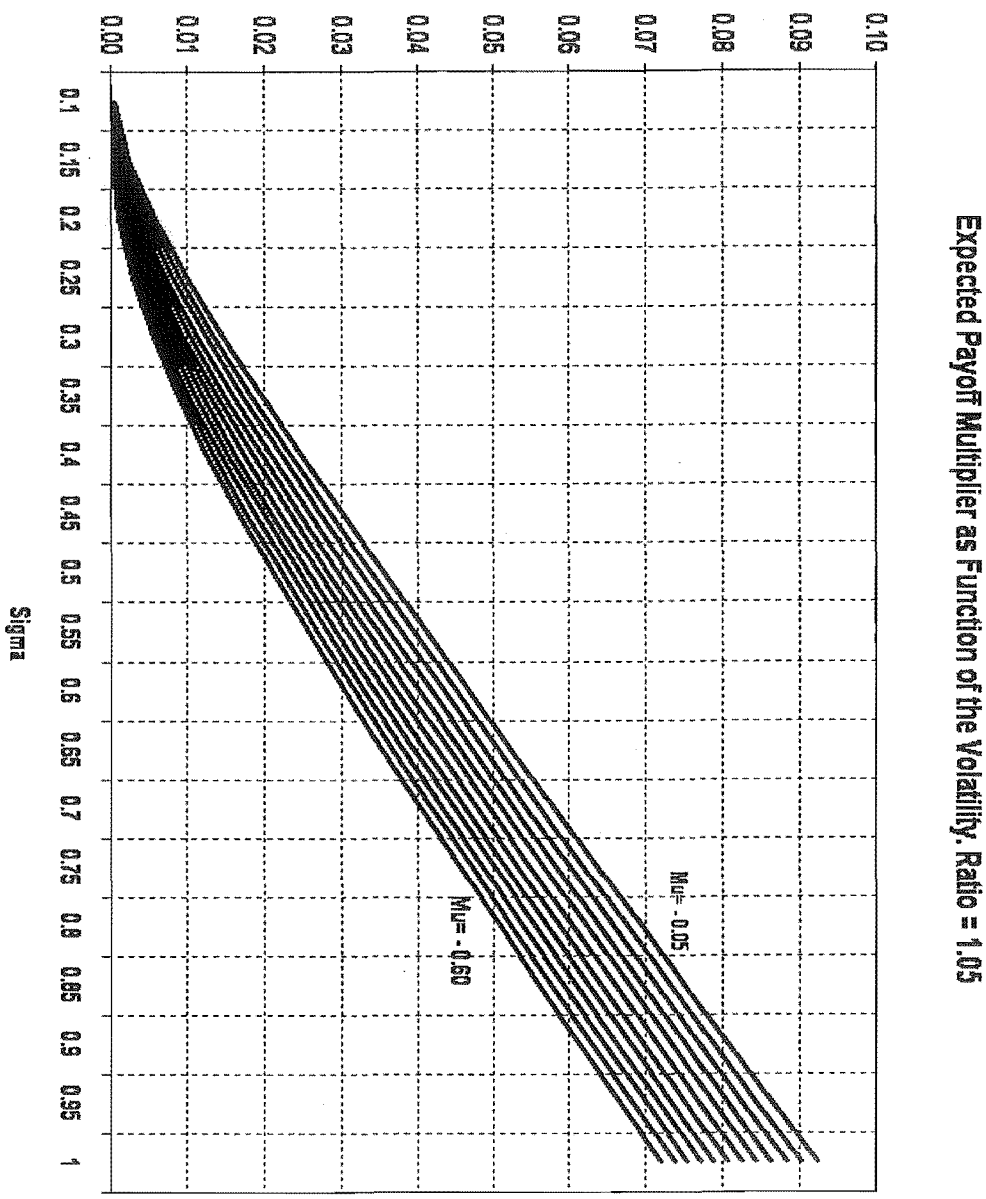


Figure 5.57: Variation of the Expected Payoff with σ (I). R= 1.05

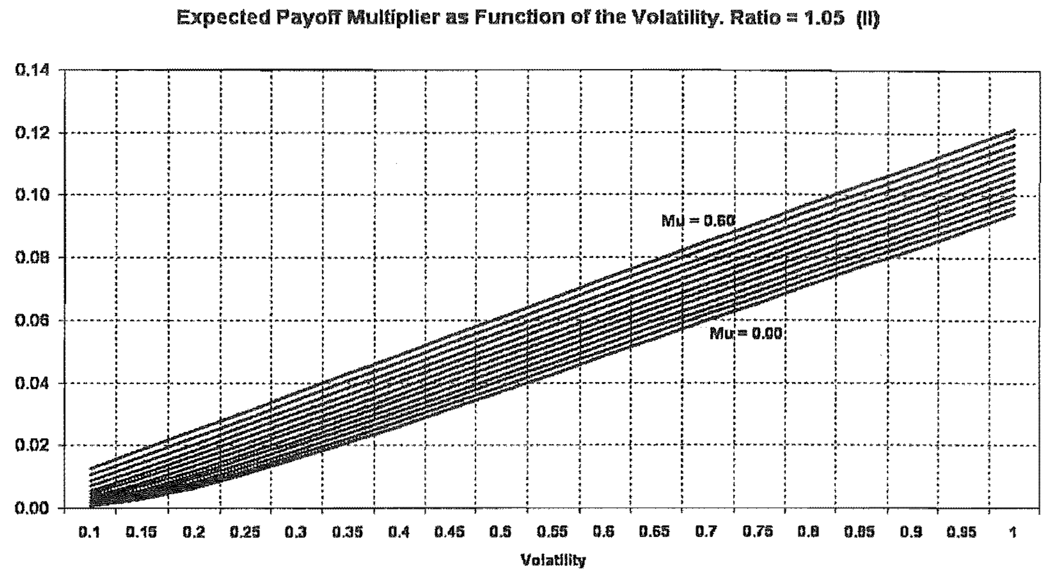


Figure 5.58: Variation of the Expected Payoff with σ (II). R= 1.05

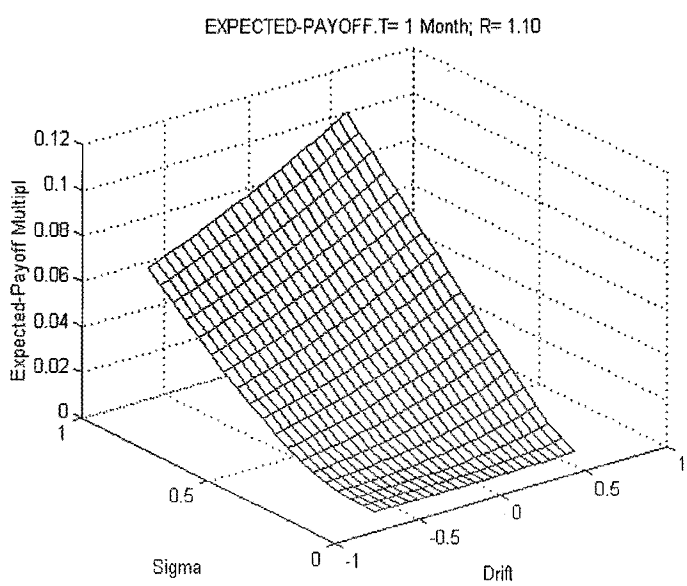


Figure 5.59: Expected Payoff Multipliers, R= 1.10

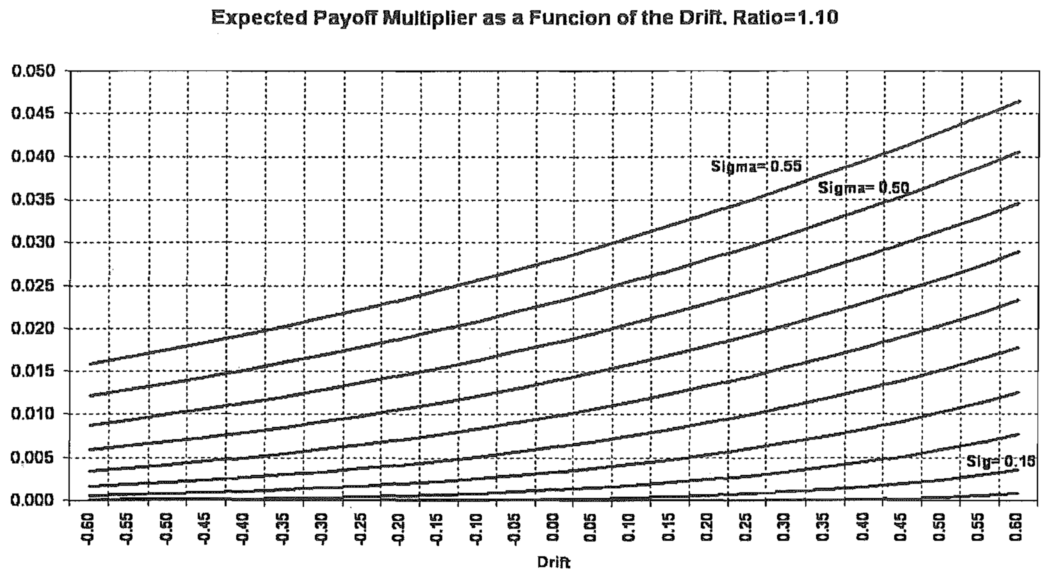


Figure 5.60: Variation of the Expected Payoff with μ (I). R=1.10

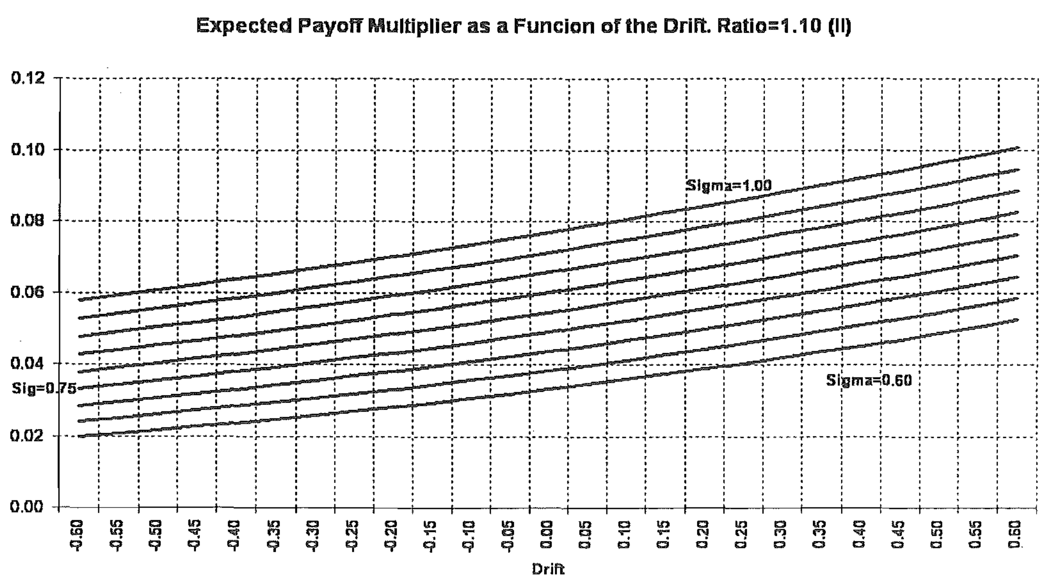


Figure 5.61: Variation of the Expected Payoff with μ (II). R= 1.10

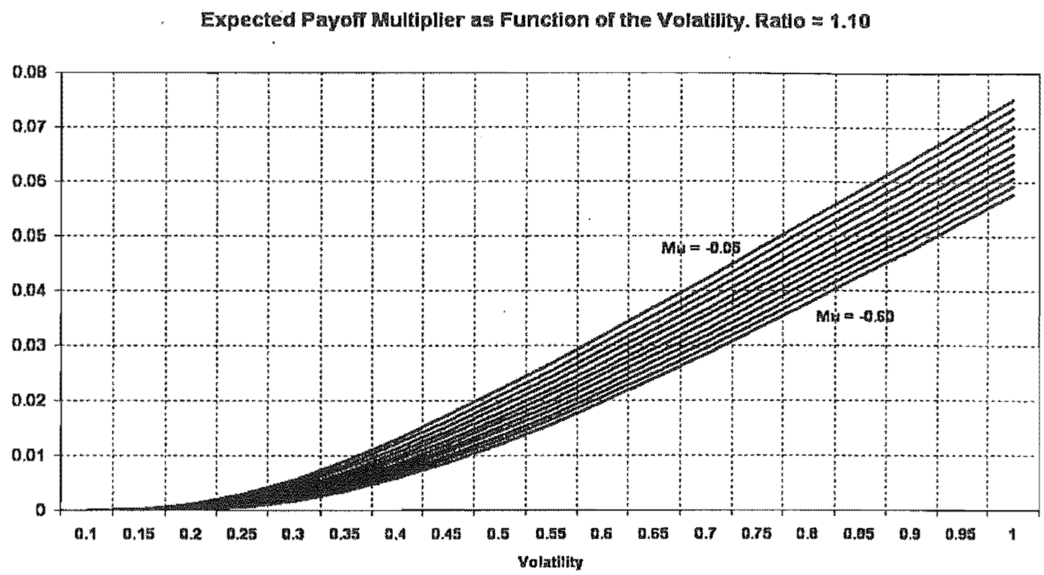


Figure 5.62: Variation of the Expected Payoff with σ (I). R=1.10

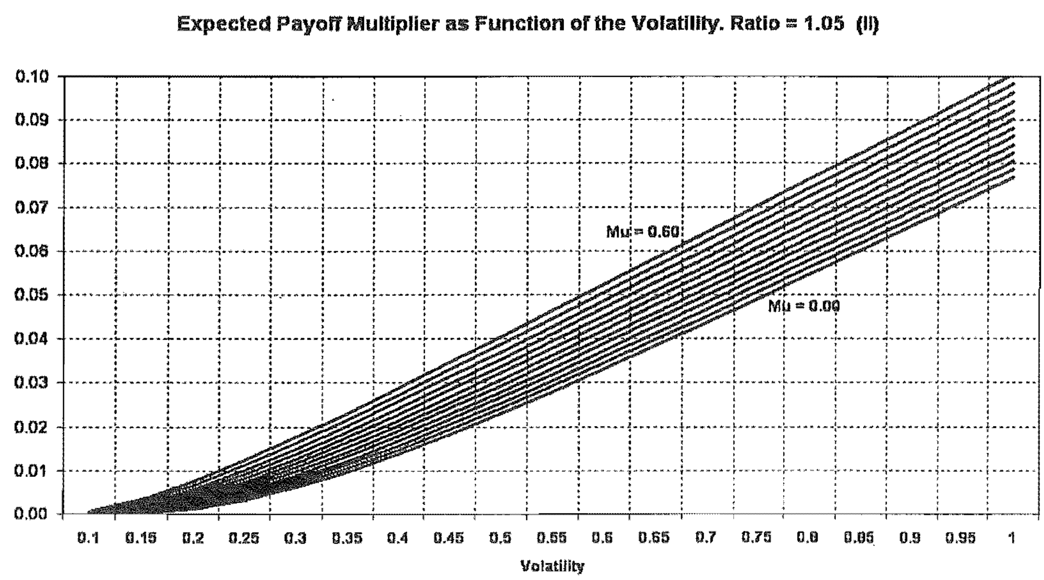


Figure 5.63: Variation of the Expected Payoff with σ. (II). R= 1.10

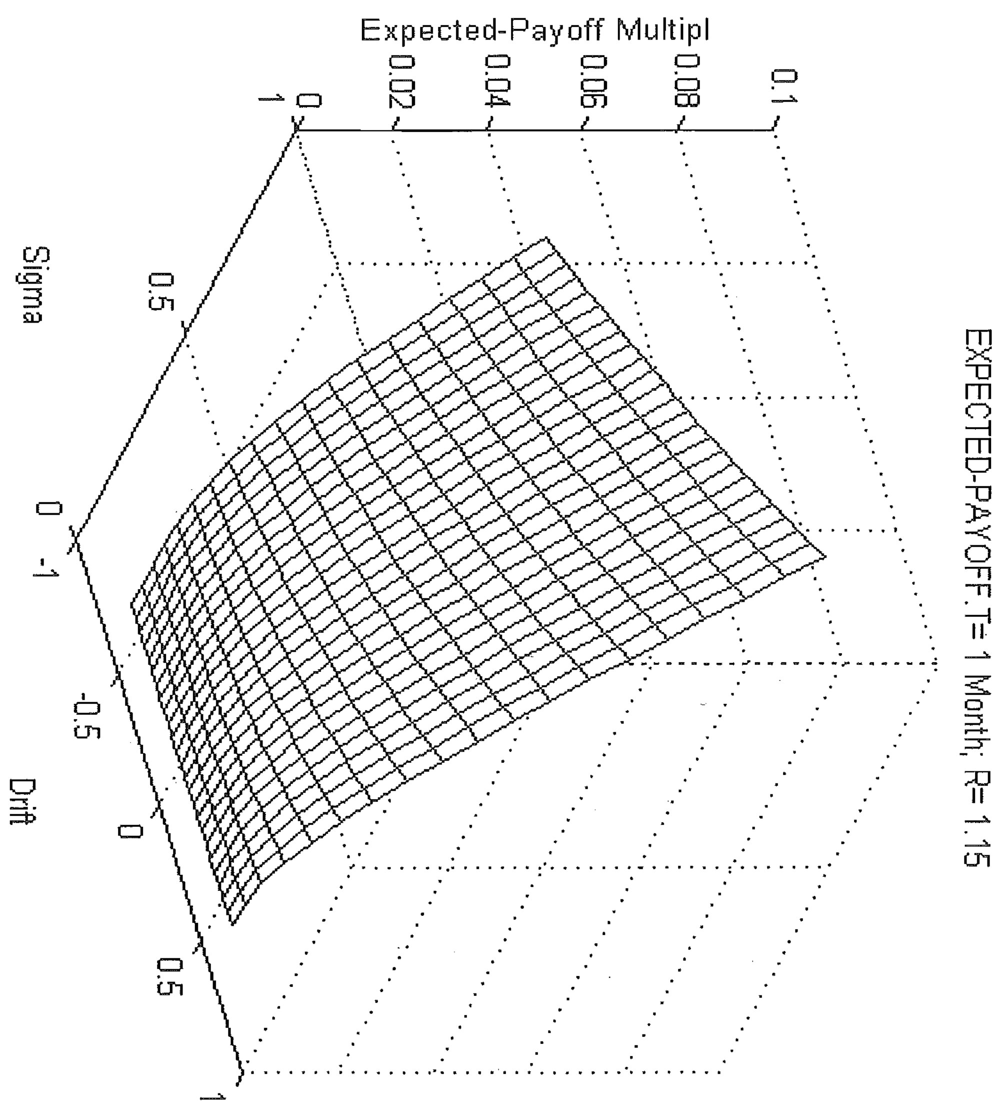


Figure 5.64: Expected Payoff Multipliers, R= 1.15

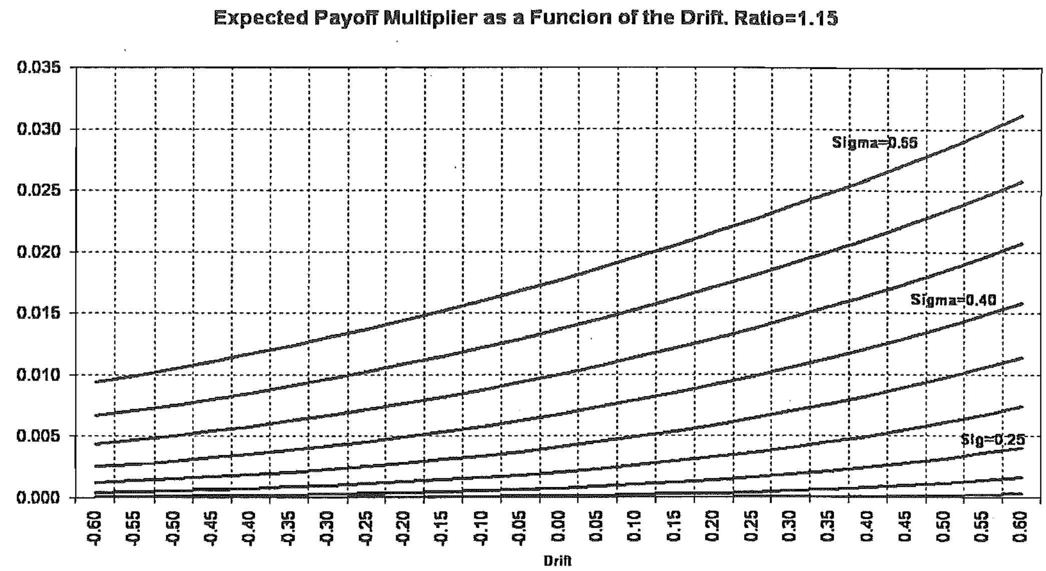


Figure 5.65: Variation of the Expected Payoff with μ (I). R= 1.15

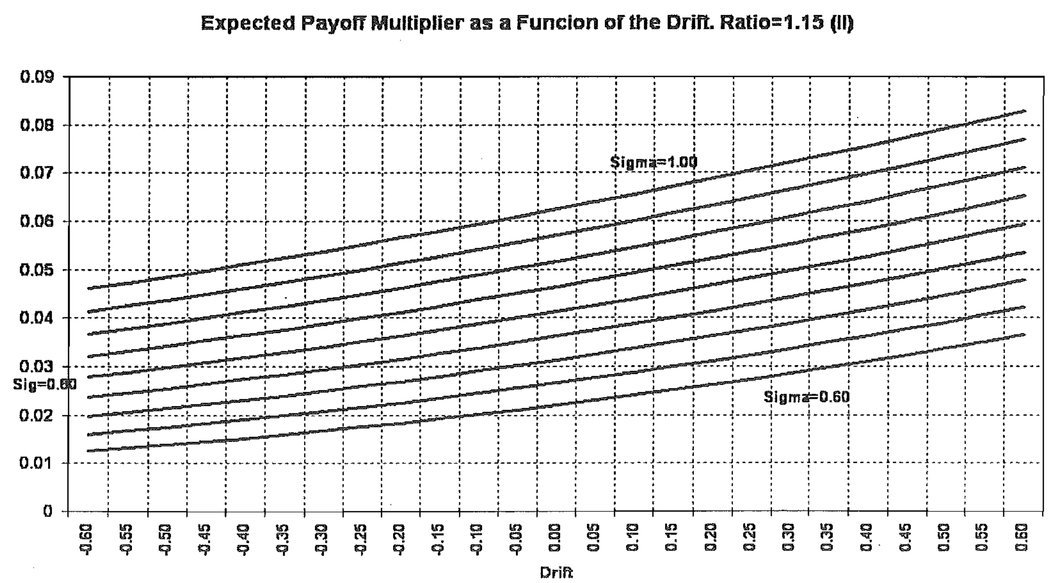


Figure 5.66: Variation of the Expected Payoff with μ (II). R= 1.15

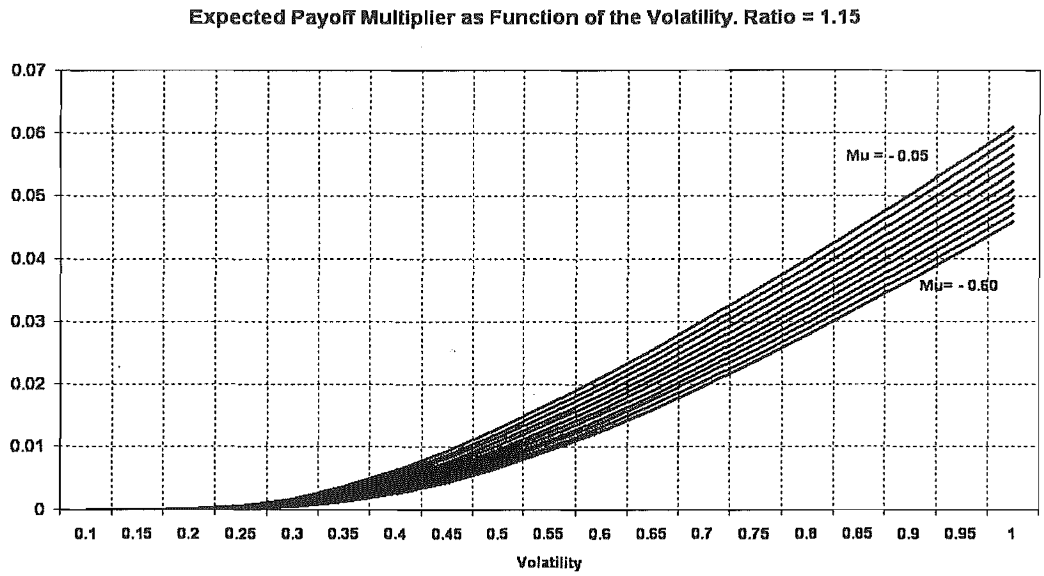


Figure 5.67: Variation of the Expected Payoff with σ (I). R= 1.15

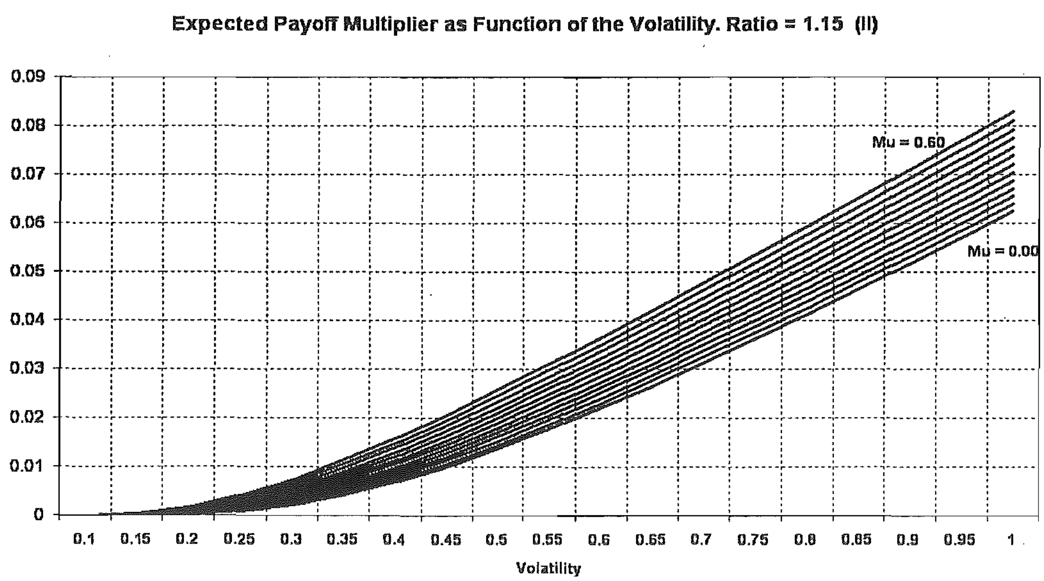


Figure 5.68: Variation of the Expected Payoff with σ (II). R= 1.15